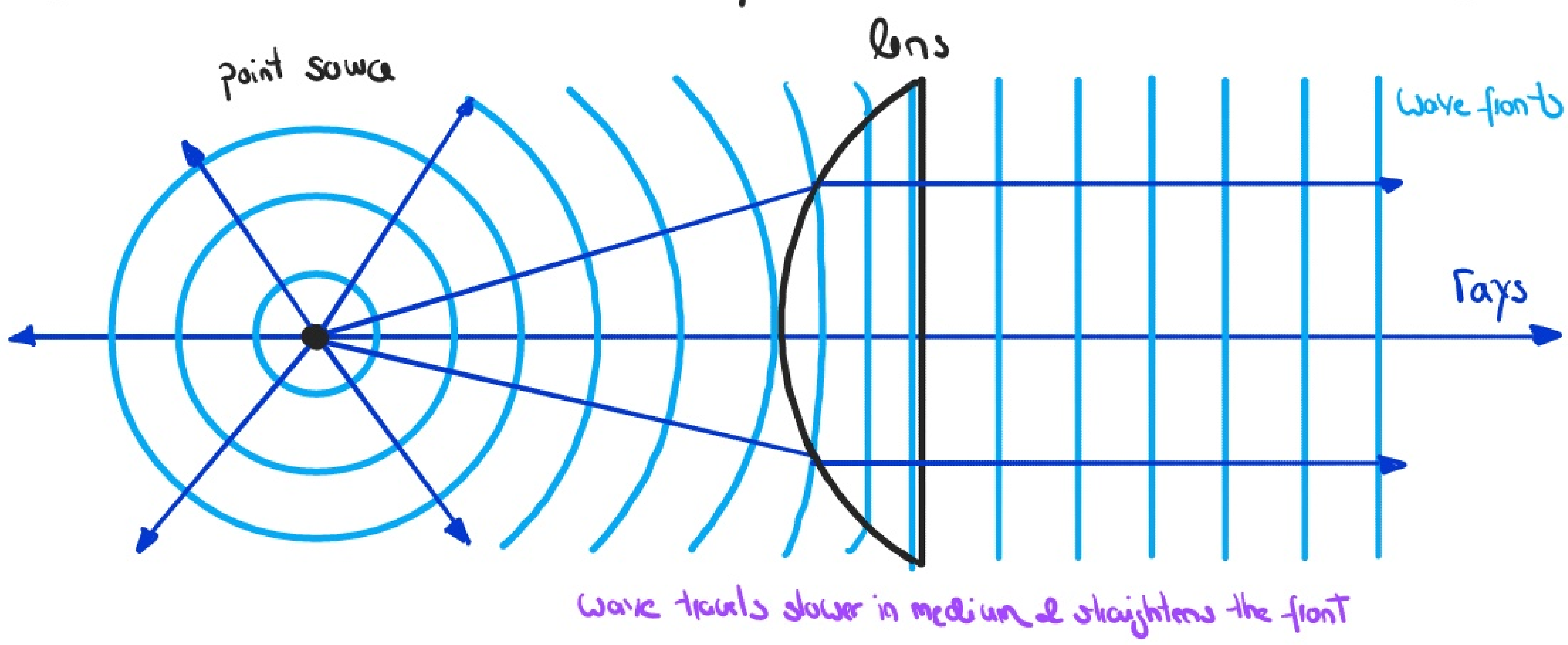
# PHYS 434 - Lecture 5

## Introduction to Lenses

### 1.) Basics

In the following, we will employ Geometric Optics to describe the propagation of light through reflective and refractive objects. This approximate method neglects diffraction, which is valid as long as the wavelength of the EM wave is much smaller than the length scales of the optical system. In this limit, the laws of reflection and refraction will be incredibly useful. We will typically go one step further and only consider rays close to the optical axis. In this paraxial approximation, incidence angles will be small and we approximate $\sin\theta \approx \tan\theta \approx \theta$.

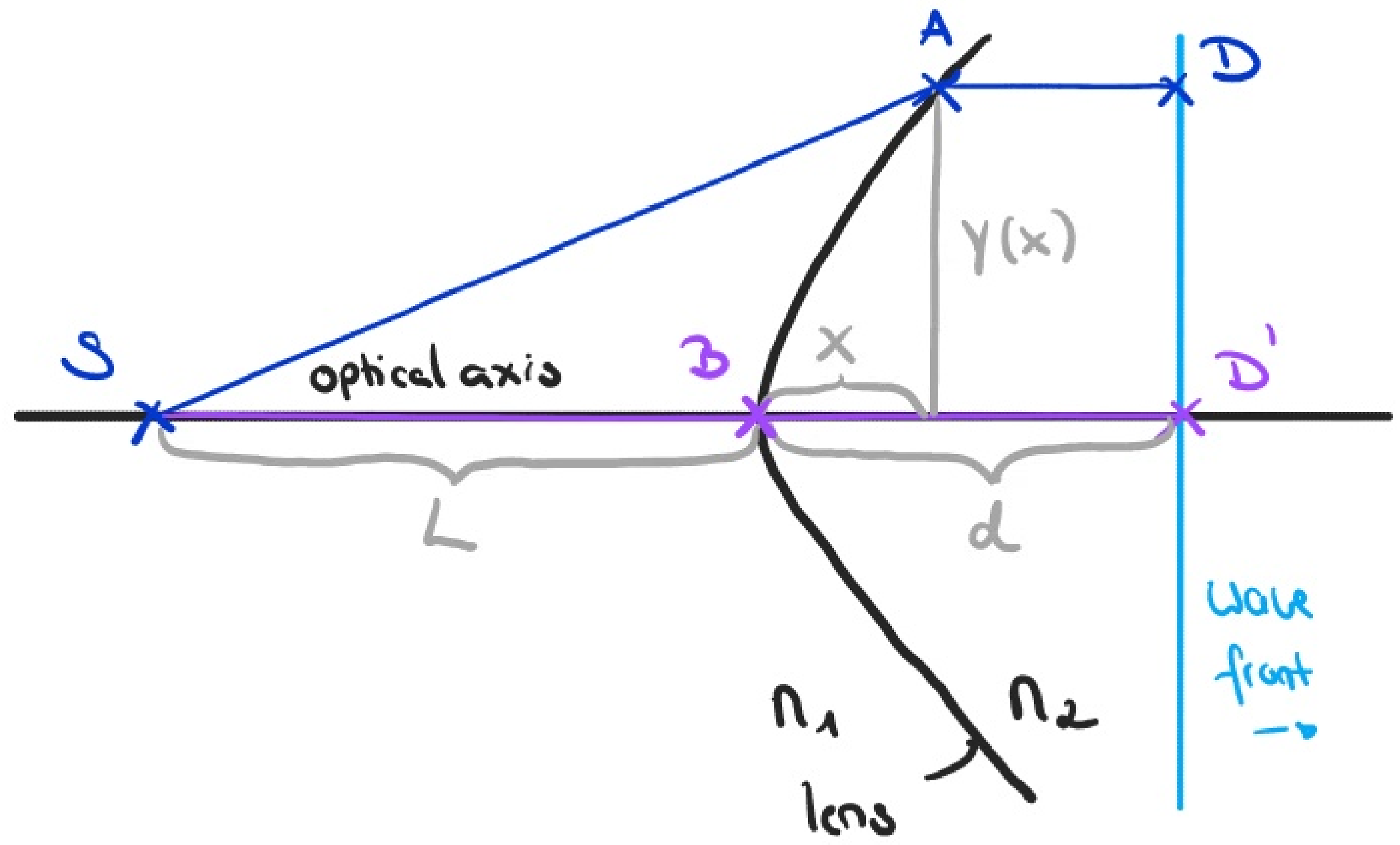
The lens is the most widely used optical device and in most general terms can be described as a refractive device (a discontinuity in the medium) that reconfigures the transmitted energy distribution. The lens changes the angular



distribution of rays. Consider for example a point source, emitting light in all directions. As the spherical wave expands, the beam weakens. That is typically not desired, so we can insert a suitable refractive medium to bend the light rays and have them emerge as a parallel wave front. Inversely, we can use a lens to collect incoming parallel rays and focus them on a single point to collect the light's energy.

## 2.) Ideal lenses

To illustrate this, we determine the shape of a lens that perfectly straightens the wave fronts from a point source. To achieve a flat wave front within the lens, the optical path length through point A (which can be anywhere on the interface) to D must be equal to the OPL $\overline{SD'}$ along the optical axis. This ensures that the phase stays the same along the wave front.




We can characterise the shape of the lens mathematically by a function $y(x)$ as shown in the sketch. This gives:

$$\overline{SD} = \overline{SD'} \rightarrow \overline{SA} + \overline{AD} = \overline{SB} + \overline{BD'}.$$

$$\Longrightarrow \quad n_1 \sqrt{y^2 + (L+x)^2} + n_2(\cancel{d} - x) = n_1 L + n_2 \cancel{d} .$$

not surprisingly $d$ drops out !!

With a bit of algebra, we find

$$\sqrt{(L+x)^2 + y^2} = L + {}^{n_2}/_{n_1} x \quad \Longrightarrow \quad (L+x)^2 + y^2 = \left(L + \frac{n_2}{n_1} x\right)^2 ,$$

$$\Longrightarrow \quad \cancel{L^2} + 2Lx + x^2 + y^2 = \cancel{L^2} + 2 \frac{n_2}{n_1} x L + \frac{n_2^2}{n_1^2} x^2 ,$$

$$\Longrightarrow \quad y^2 + x^2\left(1 - \frac{n_2^2}{n_1^2}\right) + 2Lx\left(1 - \frac{n_2}{n_1}\right)$$

$$= y^2 + \left(1 - \frac{n_2^2}{n_1^2}\right)\left[x^2 + x \frac{2L(1 - {}^{n_2}/_{n_1})}{1 - {}^{n_2^2}/_{n_1^2}}\right]$$

$(a^2 - b^2) = (a+b)(a-b)$

$$= y^2 + \left(1 - \frac{n_2^2}{n_1^2}\right)\left[x^2 + x \frac{2L}{1 + {}^{n_2}/_{n_1}}\right]$$

add 0

$$= y^2 + \left(1 - \frac{n_2^2}{n_1^2}\right)\left[x + \frac{L}{1 + {}^{n_2}/_{n_1}}\right]^2 - \frac{1 - {}^{n_2^2}/_{n_1^2}}{(1 + {}^{n_2}/_{n_1})^2} L^2 ,$$

$$\Longrightarrow \quad y^2 + \left(1 - \frac{n_2^2}{n_1^2}\right)\left[x + \frac{L}{1 + {}^{n_2}/_{n_1}}\right]^2 = \frac{1 - {}^{n_2}/_{n_1}}{1 + {}^{n_2}/_{n_1}} L^2 .$$

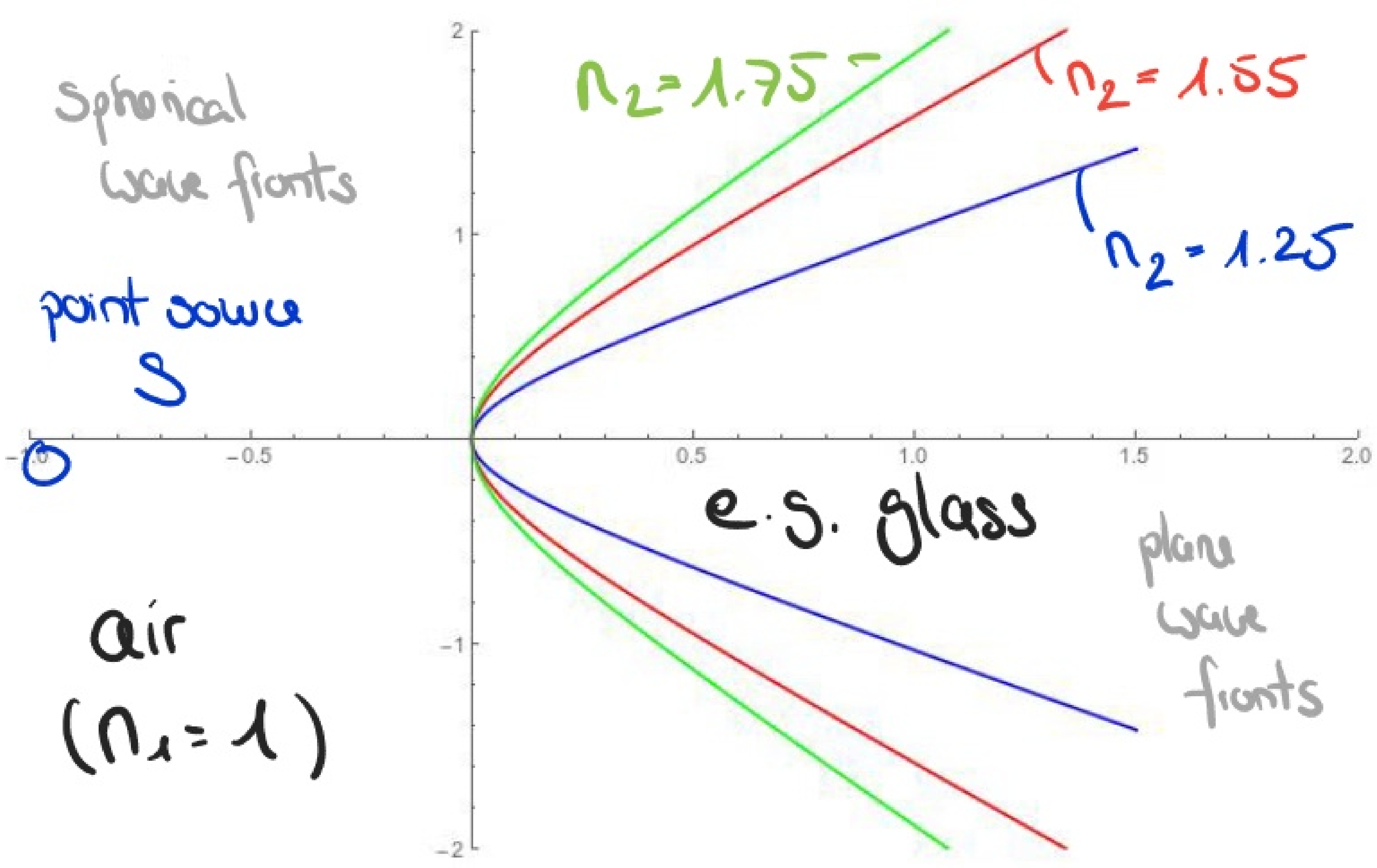
We can write this in the following form

$$\underline{\underline{\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1}} \quad \text{with} \quad x_0 = \frac{-L}{1 + {}^{n_2}/_{n_1}} ,$$

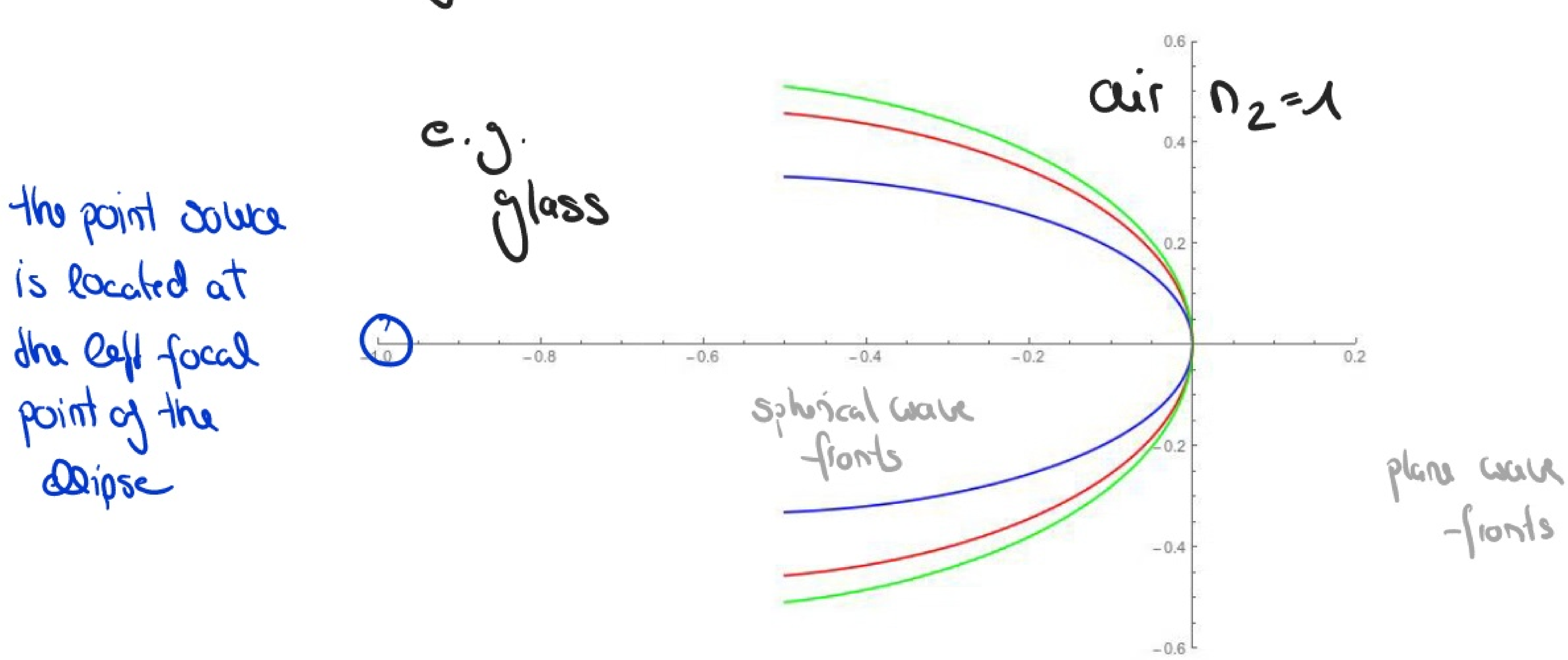
$$a^2 = \frac{L^2}{(1 + {}^{n_2}/_{n_1})^2} \quad \text{and} \quad b^2 = \frac{1 - {}^{n_2}/_{n_1}}{1 + {}^{n_2}/_{n_1}} L^2 .$$

This is the equation for an ellipse if $b^2 > 0$ and a hyperbola if $b^2 < 0$. Depending on the refractive indices, we thus require a hyperbolic lens shape to straighten a wave front if $n_2 > n_1$ and an elliptic shape for $n_1 > n_2$.

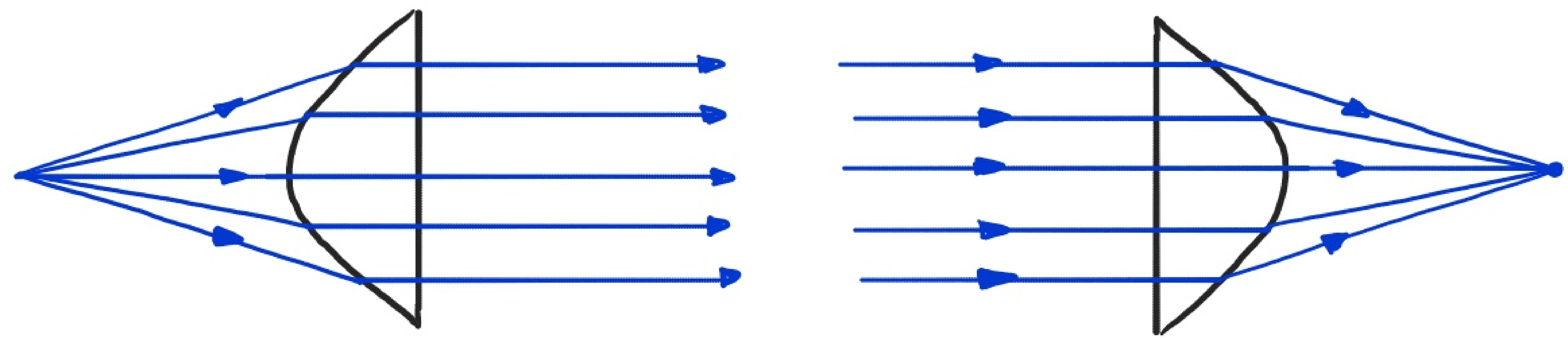
Hyperbolic lens profiles for $n_1 = 1$ and three different $n_2$ values look for example like this, where a smaller ratio $n_2/n_1$ corresponds to a flatter lense; i.e. the larger the index contrast the less curved the surface needs to be, because the wave front in the lens material is slowed down quicker.



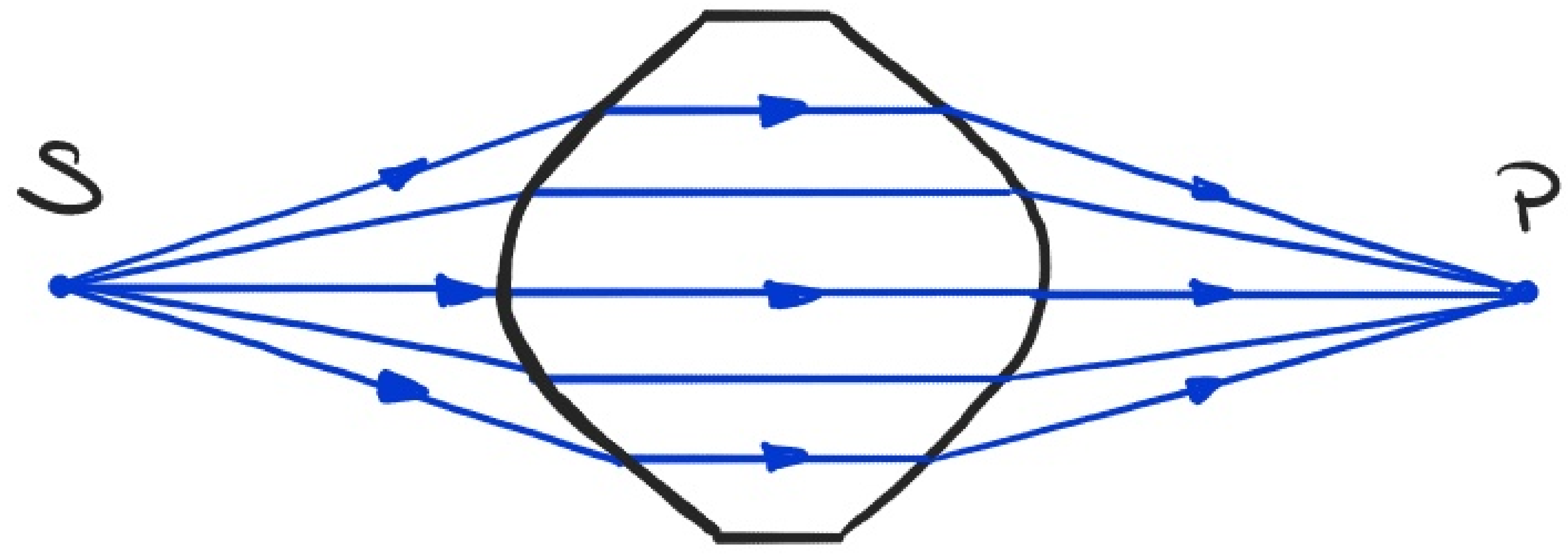

Or in the case of the elliptic lens profile



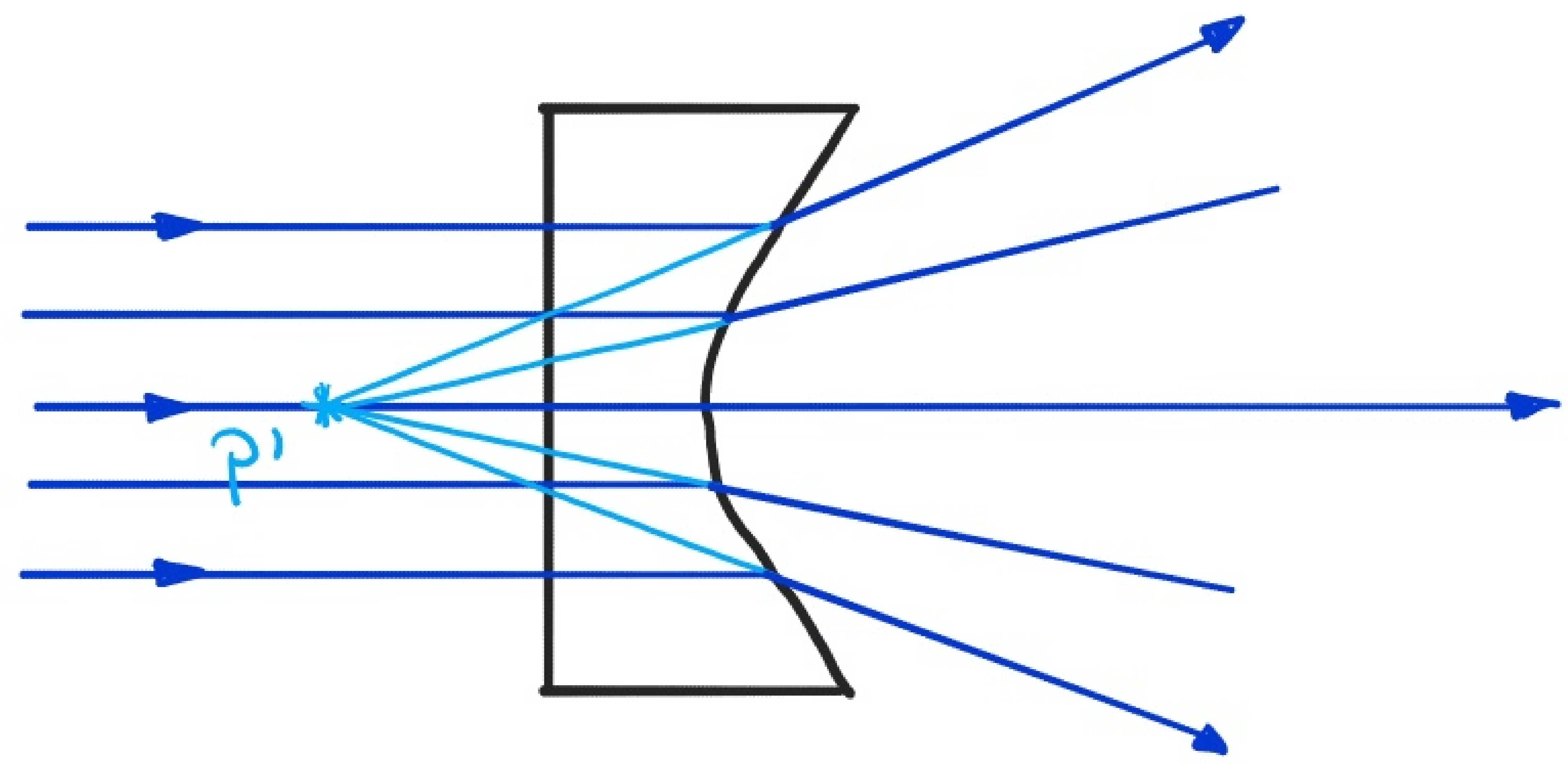
Because the path of light has to be reversible, we conclude that a lens of specific shape can not only convert spherical wave-fronts into plane ones, but the same shape will also focus a plane wave front onto one point, the focal point of the lens.



We can use this concept to combine these two surfaces to a convex (from Latin, meaning arched) lens, which can be used to create a 'real' image at a point P of the source positioned at S. The two points are referred to as conjugate points.



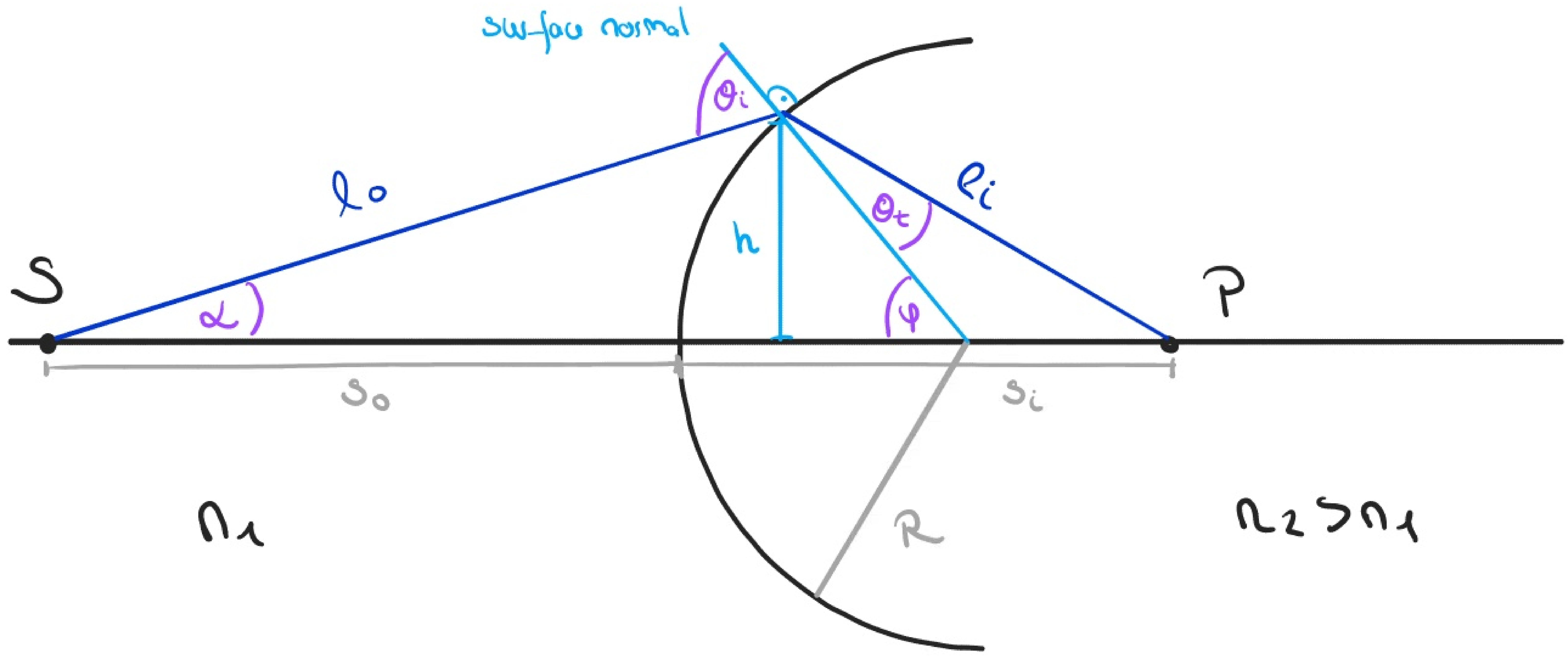

In contrast, a concave (from Latin, meaning hollow) lens does not converge the rays, but causes divergence instead. Such a lens has a hyperbolic profile and the diverging rays seem to come from a focal point P'. As no real source exists at this point, we refer to the observed image as a 'virtual' image (see next page).




## 3.) Spherical lenses

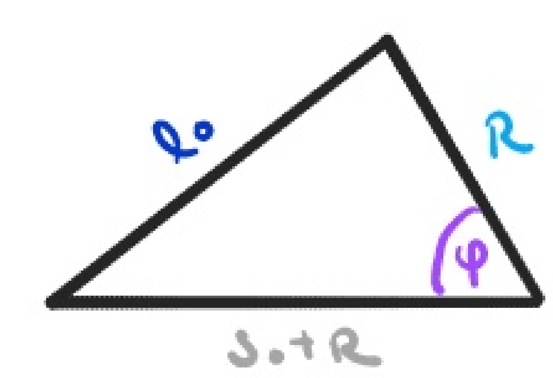
While optimal shapes to focus and collimate light are aspheric, spherical lenses are much easier and cheaper to manufacture because of their constant radius of curvature. Thus, a large number of lenses are segments of spheres and we will discuss their properties below. Questions that we have to address are (i) When can we use spherical elements to almost perfectly collimate and focus light? (ii) What are the imaging properties of spherical elements in this regime? and (iii) What imperfections (also called aberrations) occur from using spherical surfaces? We will discuss the first two points now and aberrations at a later instance (see Lecture 9).

Consider a point source S emitting spherical wavefronts that are refracted off a spherical surface with curvature radius R (see geometry on the next page). We want to know, if the source is a distance $s_o$ in front of the surface, at what distance $s_i$ will the ray again intersect with the optical axis. Note that $s_o$ and $s_i$ are commonly referred to as object and image distance.

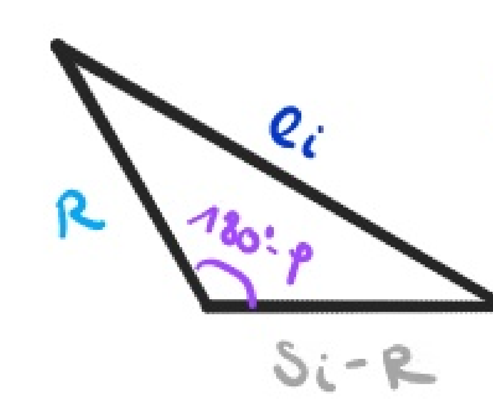



To answer this question, we have to calculate the respective optical path lengths and apply Fermat's principle, varying the point at which the ray connecting S and P intersects the spherical surface. Looking at the two triangles and recalling the law of cosines, we can express $\ell_o$ and $\ell_i$ as

$$\ell_o^2 = R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\varphi ,$$




$$\ell_i^2 = R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\varphi .$$




$\cos(180° - \varphi) = -\cos\varphi$

The optical path length in terms of the fixed quantities $s_o$, $s_i$ and $R$ is thus given by

$$\begin{aligned} OPL &= n_1 \ell_o + n_2 \ell_i \\ &= n_1 \left[R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\varphi\right]^{1/2} \\ &\quad + n_2 \left[R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\varphi\right]^{1/2} . \end{aligned}$$

Fermat's principle maintains that the OPL will be stationary. Taking the derivative with respect to $\varphi$ thus dictates ( $\partial OPL/\partial\varphi = 0$ ):

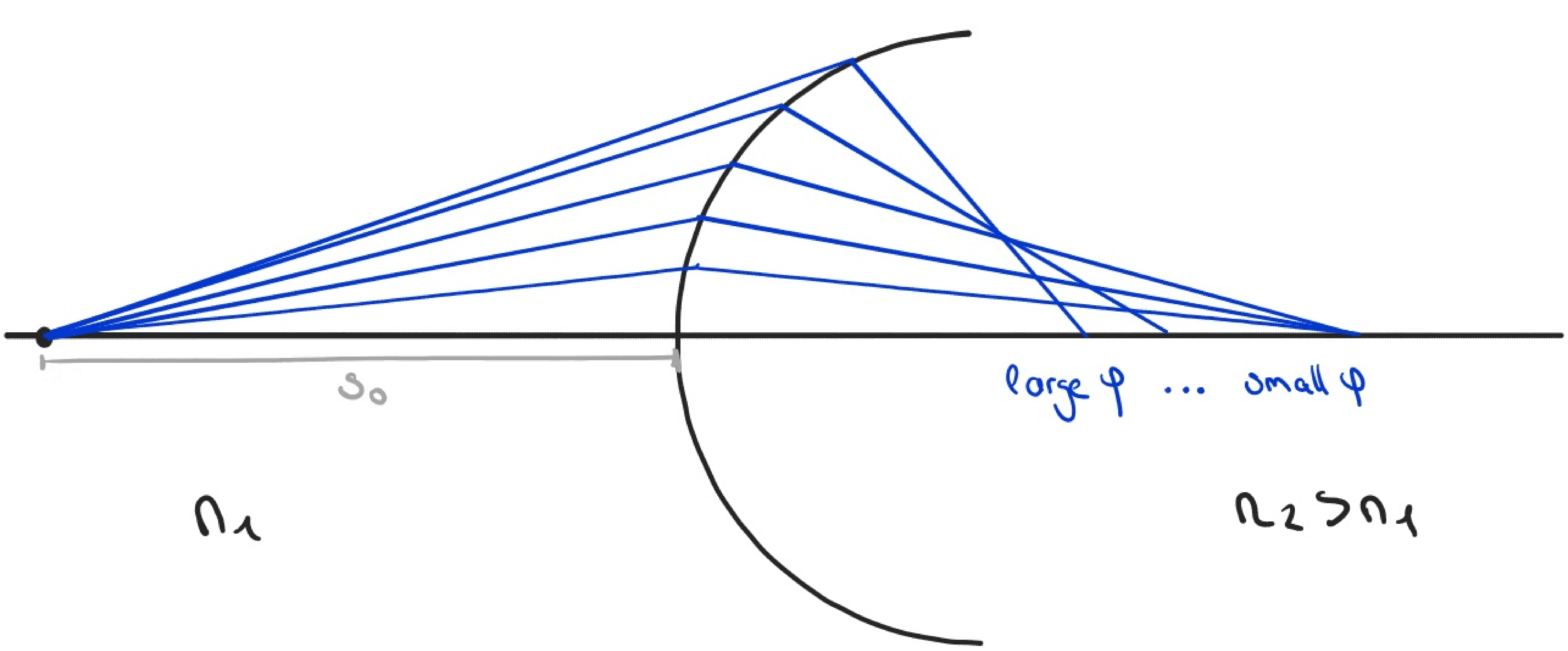
$$n_1 \frac{\partial l_o}{\partial \varphi} = - n_2 \frac{\partial l_i}{\partial \varphi} ,$$

$$\Rightarrow \quad n_1 \frac{(-\cancel{2}R)}{\cancel{2}l_o}(s_o+R)(-\cancel{\sin\varphi}) = -n_2 \frac{\cancel{2}R}{\cancel{2}l_i}(s_i-R)(-\cancel{\sin\varphi}),$$

$$\Rightarrow \quad n_1 (s_o+R)/l_o = n_2 (s_i-R)/l_i ,$$

$$\Rightarrow \quad \frac{n_1}{l_o} + \frac{n_2}{l_i} = \frac{1}{R}\left(\frac{n_2 s_i}{l_i} - \frac{n_1 s_o}{l_o}\right) .$$

Although this equation is exact, it is quite complicated. Specifically note that $l_o$ and $l_i$ depend on $\varphi$. This implies that $s_i$ not only depends on $R$, $n_1$, $n_2$ and $s_o$ but also $\varphi$. Hence, the rays emanating from the source at different angles (which corresponds to different angles $\varphi$) will cross the optical axis at different points $P$.



# 4.) Paraxial approximation

As observed in the last sketch, for small $\varphi$, many rays land at almost the same point. Often, we are exactly interested in the directions of light propagation close to the optical axis. For sufficiently small angles between the rays and the optical axis, we can approximate the trigonometric functions as

$$\cos\varphi \approx 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots ,$$

$$\sin\varphi \approx \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots .$$

For small angles, we can stop at the first order term (the higher order terms will play a role for aberrations). In this limit, spherical lenses become very useful for collimating and focussing light. Mathematically, we obtain for $\varphi \ll 1$ that

$$l_o \approx \left(R^2 + (s_o+R)^2 - 2R(s_o+R)\right)^{1/2} = s_o ,$$

$$l_i \approx \left(R^2 + (s_i-R)^2 + 2R(s_i-R)\right)^{1/2} = s_i .$$

For a single spherical surface, we thus arrive at

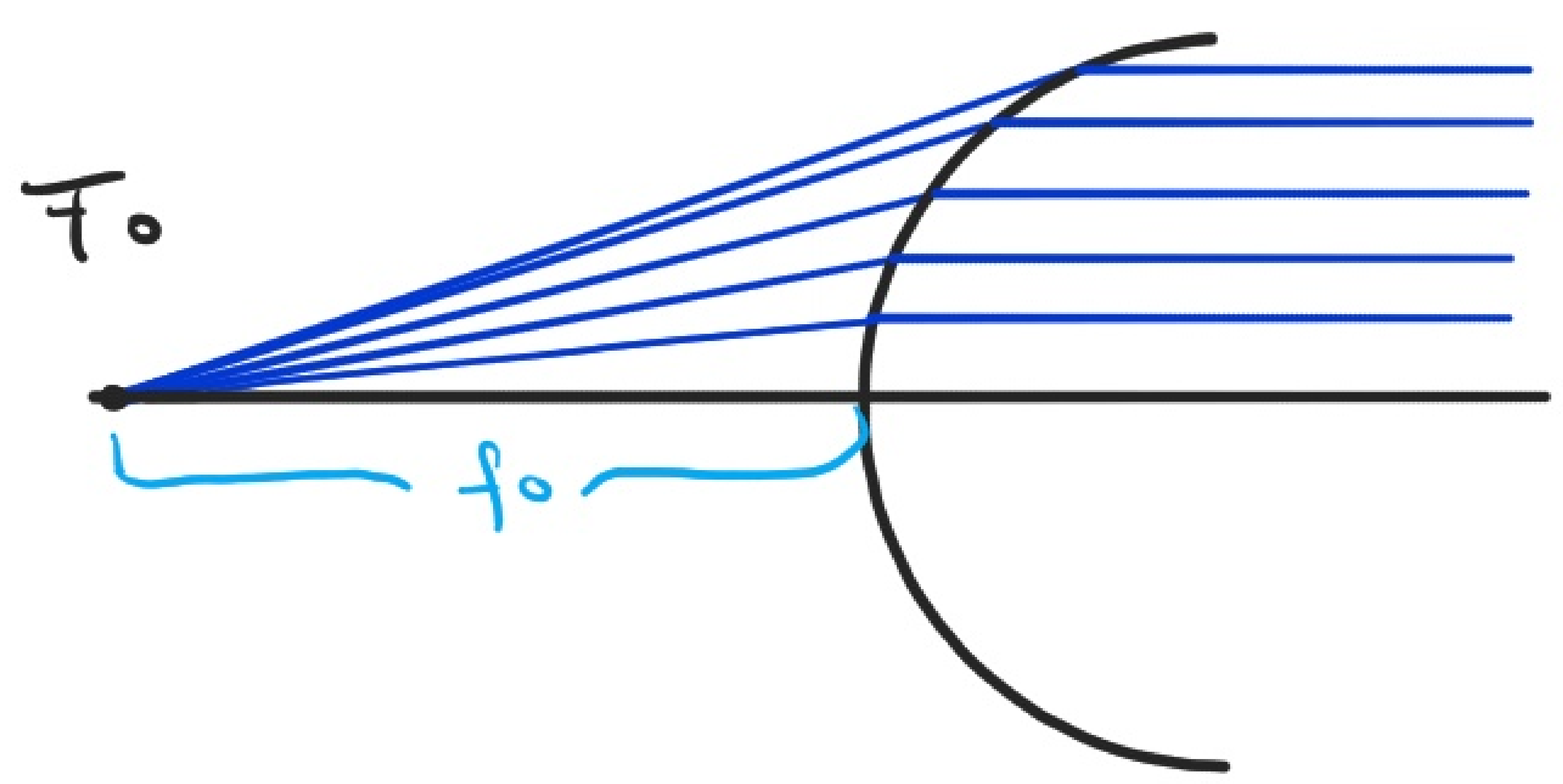
This first-order theory was first developed by Gauss.

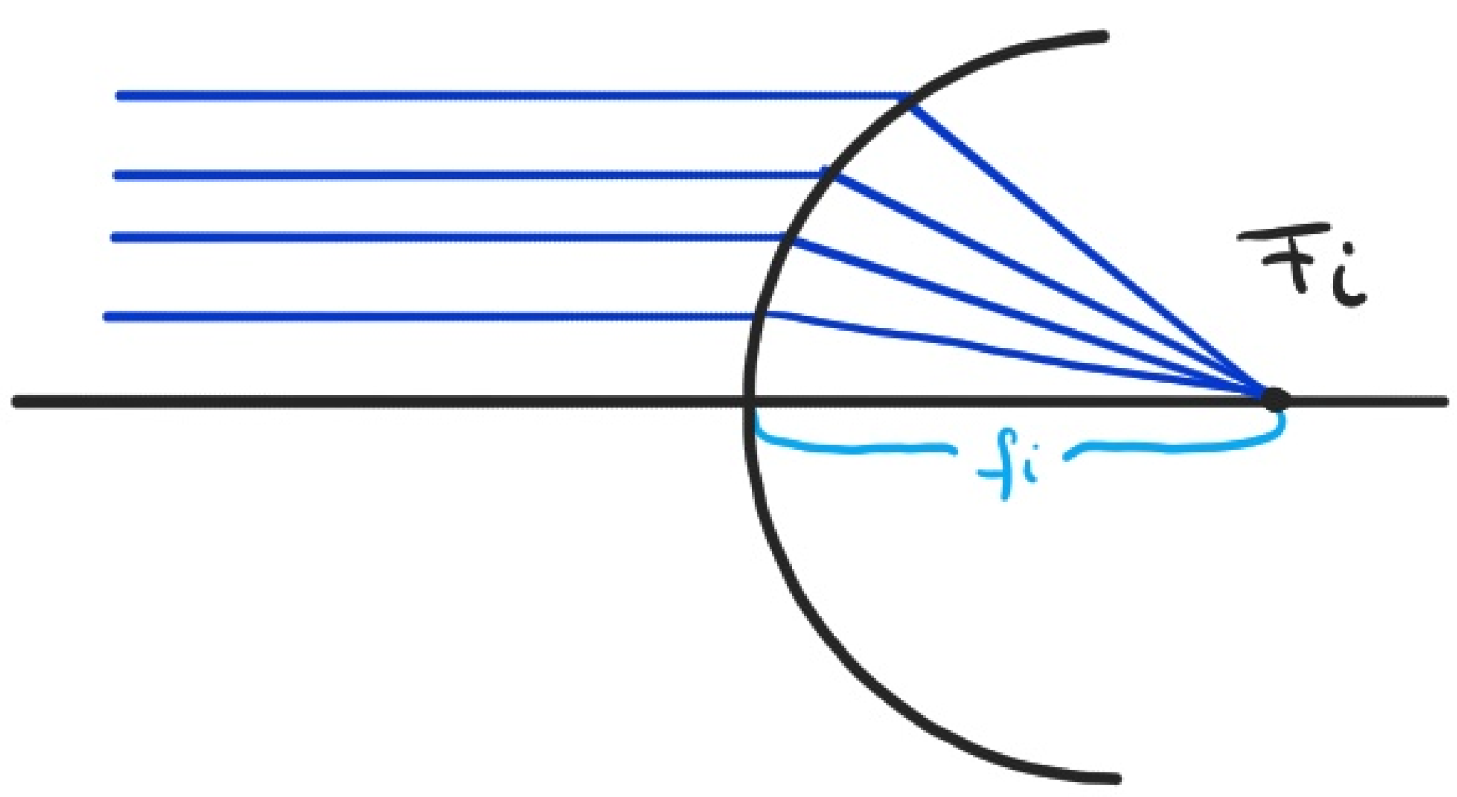
$$(*) \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} .$$

We can use this equation to describe the collimation (left sketch) and focussing (right sketch) of light. If the object is imaged at infinity ($s_i \to \infty$) or the image is coming from infinity ($s_o \to \infty$), we have

$$\frac{n_1}{s_o} = \frac{n_2 - n_1}{R} \qquad \Big| \qquad \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

$$\Rightarrow s_o = f_o = R\,\frac{n_1}{n_2 - n_1} \qquad \Big| \qquad \Rightarrow s_i = f_i = R\,\frac{n_2}{n_2 - n_1}$$

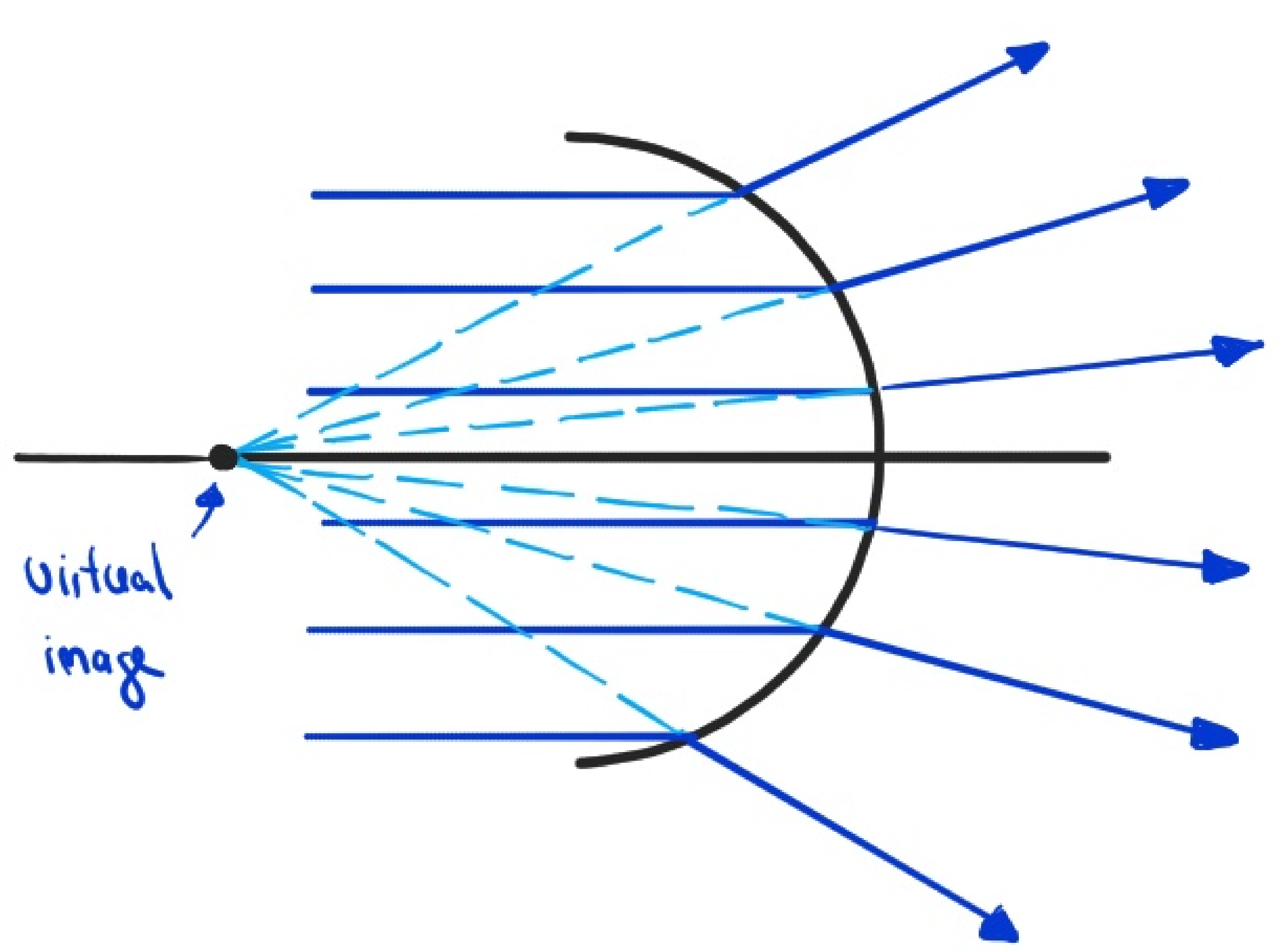



The points $F_o$ / $F_i$ are commonly referred to as the object / image focus, whereas the lengths $f_o$ / $f_i$ are called object / image focal lengths. You should now be able to answer the following questions:

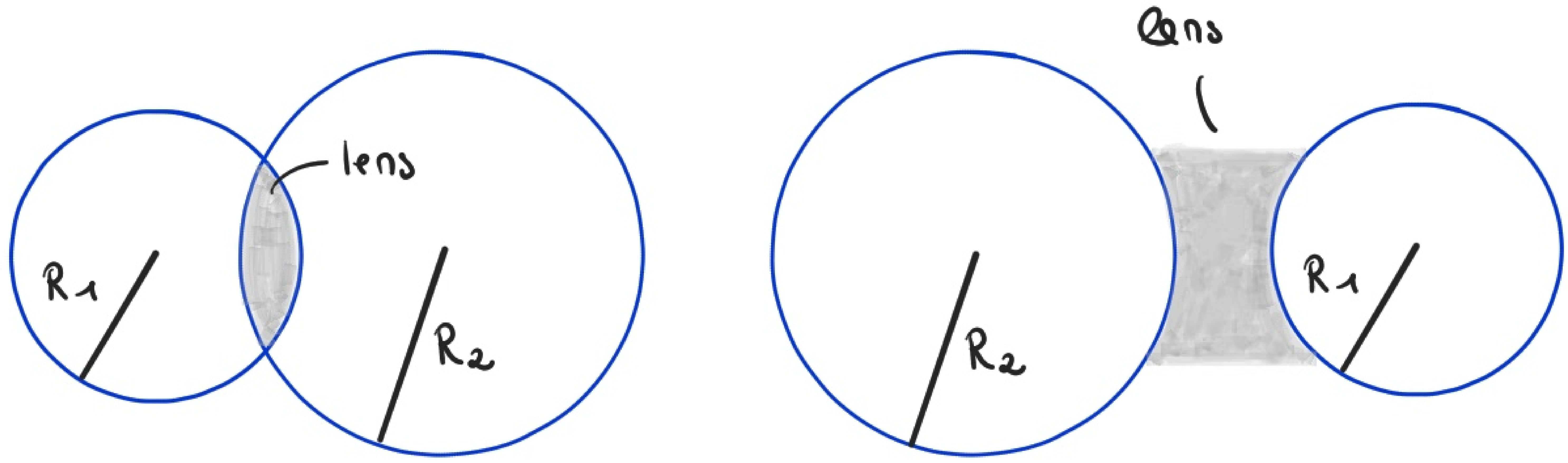
**Question 1:** What happens to the outgoing rays in the collimation scenario, when you bring the source closer to the lense / move it further away?

**Question 2:** Consider a concave spherical surface illuminated by a plane wave. Where will the virtual image be visible?



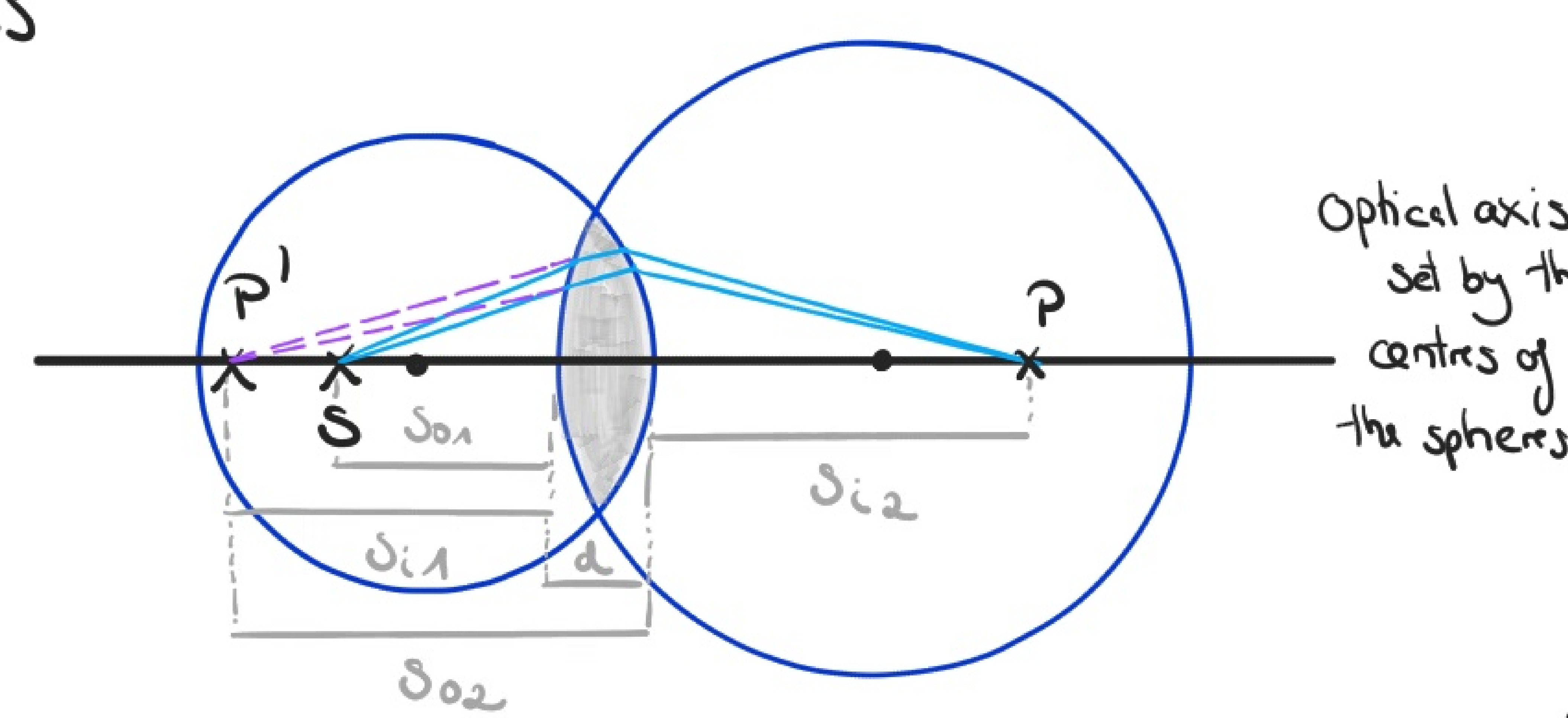
# 5.) Thin lenses

Lenses are generally made of two or more refracting interfaces, of which at least one is curved, and the nonplanar surfaces are centered on a common axis. The simplest lenses are formed by two spherical surfaces, which can be convex ($R$ is positive), concave ($R$ is negative) or flat ($R \to \infty$), e.g.




We focus on the left sketch and want to determine the position of the two conjugate points $S$ and $P$ for a lens with refractive index $n_e$ and a surrounding medium $n_m$. Consider the following situation: A source $S$ is positioned a distance $s_{o1}$ from the first surface, which satisfies $s_{o1} < f_{o1}$. The refraction off the first interface, thus, creates a virtual image of $S$ at the point $P'$. $P'$ is located at a (negative) distance $s_{i1}$ from the surface, which can be calculated from Eqn. (*) as

$$\frac{n_m}{s_{o1}} + \frac{n_e}{s_{i1}} = \frac{n_e - n_m}{R_1}.$$




The second surface 'sees'

The rays come from this point P' and refraction off the second interface focuses the rays onto point P. At the second interface the roles of $n_e$ and $n_m$ are inverted as we start out inside the lens and $R_2 < 0$ because the surface is curved to the left. Using again Eqn. (*), we arrive at

$$\frac{n_e}{s_{o2}} + \frac{n_m}{s_{i2}} = \frac{n_m - n_e}{R_2} .$$

because $|s_{o2}| = |s_{i1}| + d$ & $s_{i1} < 0$

A closer look at the geometry suggests $s_{o2} = -s_{i1} + d$, where $d$ is the thickness of the lens. In the limit of a thin lens, where $d \ll f_i$, we can approximate $s_{o2} \approx -s_{i1}$ and combine the previous two equations to

$$\frac{n_m}{s_{i2}} - \frac{n_m - n_e}{R_2} = -\frac{n_e}{s_{o2}} \overset{\text{thin lens}}{\approx} \frac{n_e}{s_i} \overset{\text{first relation}}{=} \frac{n_e - n_m}{R_1} - \frac{n_m}{s_{o1}} ,$$

$$\Rightarrow \quad \frac{1}{s_{o1}} + \frac{1}{s_{i2}} = \frac{n_e - n_m}{n_m}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) .$$

For air with $n_m \approx 1$, this reduces to the well-known 'thin-lens' or 'lensmaker's' equation, where we set $s_{o1} = s_o$ and $s_{i2} = s_i$:

$$\frac{1}{s_o} + \frac{1}{s_i} = (n_e - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) .$$

From this equation it is obvious that the $f_i = f_o = f$, so we find

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i} .$$

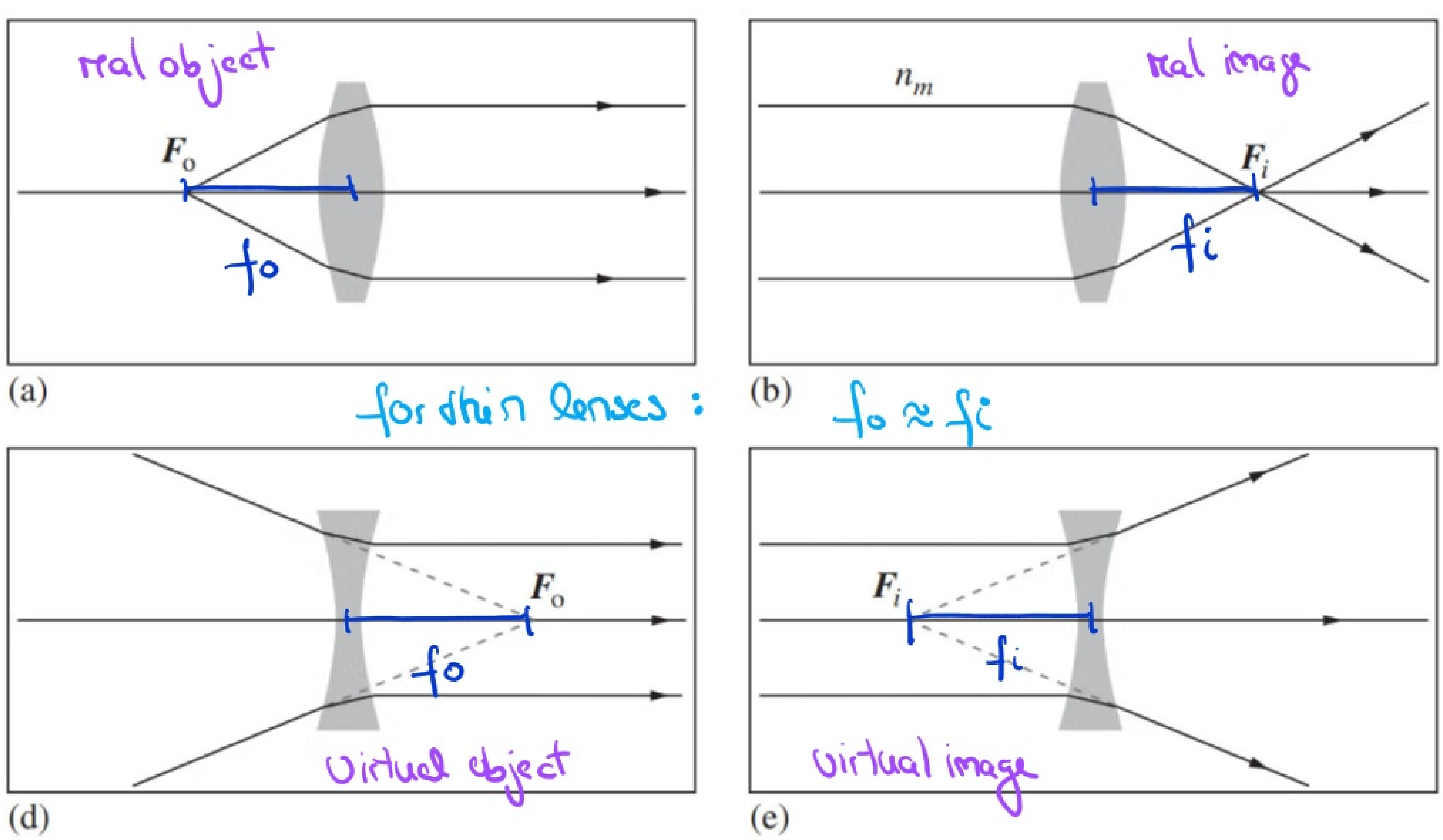
Gaussian lens formula

# PHYS 434 – Lecture 6

## Lenses, Mirrors

### 1.) Focal plane

'Lensmaker's' equation provides an analytic relation to characterise the light paths of the most common lenses. The four most common scenarios (where the focal points $F_o$, $F_i$ lie on the optical axis) are illustrated below.

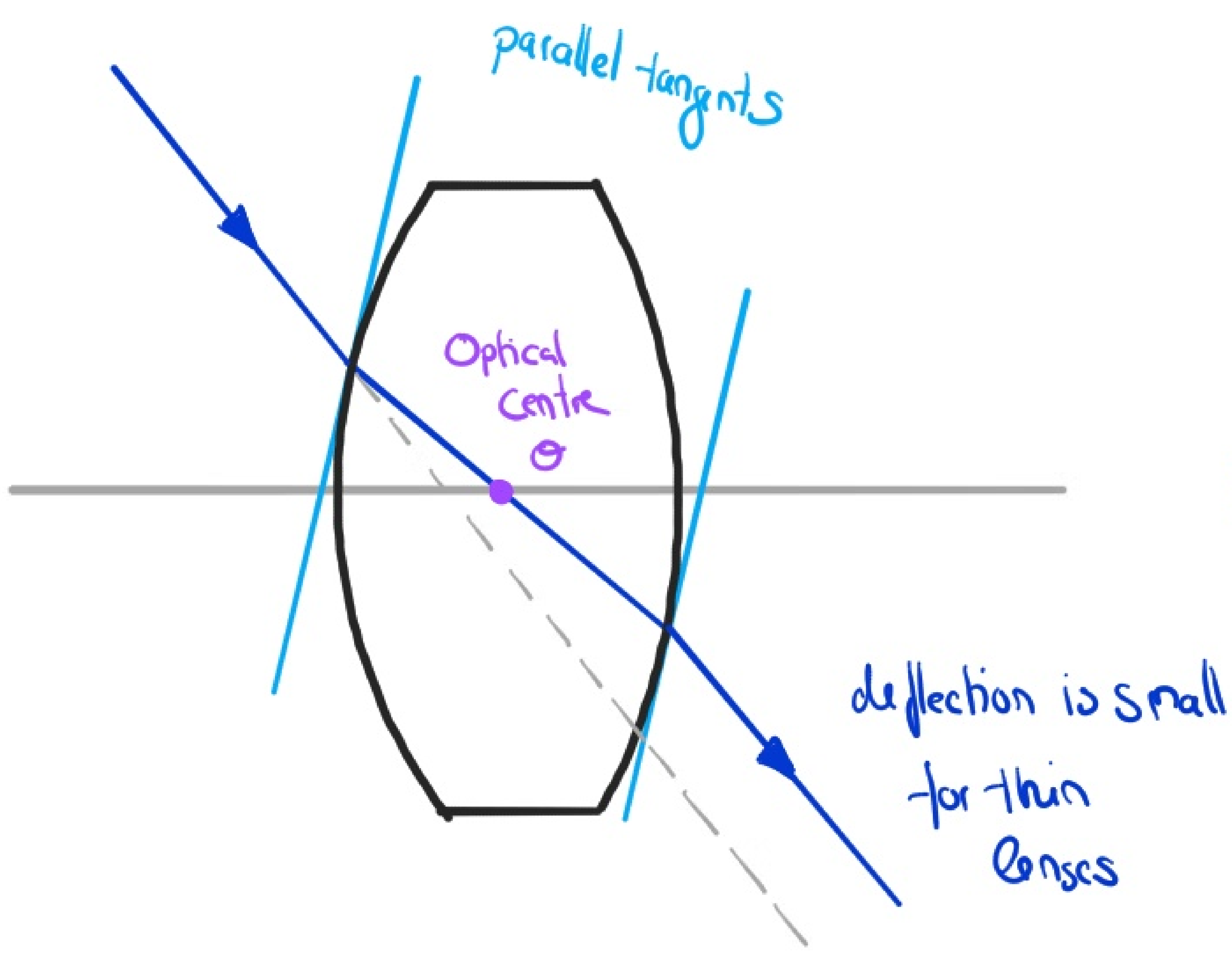


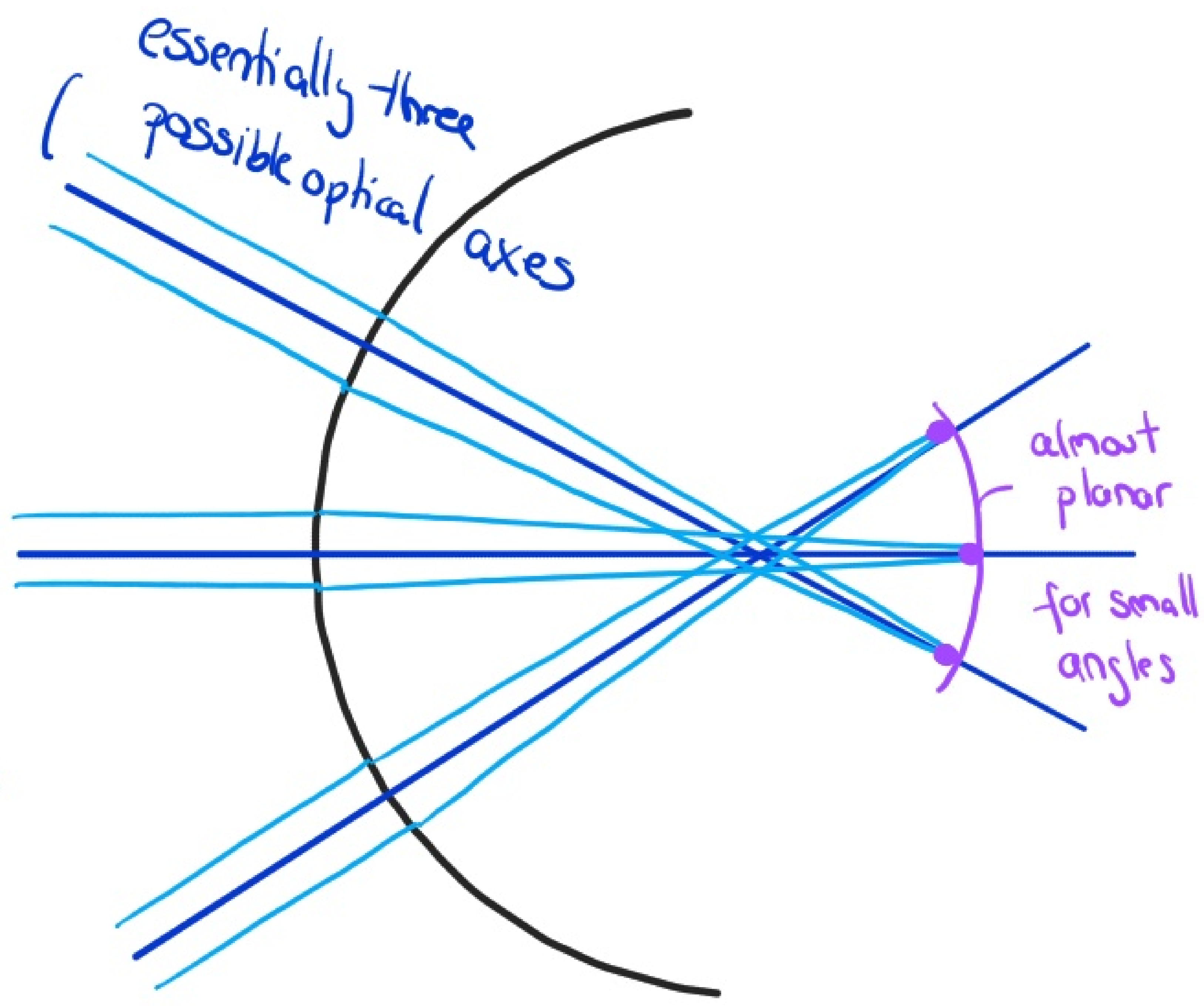

for thin lenses : $f_o \approx f_i$

$R_1 > 0$ & $R_2 < 0$ $\Rightarrow f > 0$

$R_1 < 0$ & $R_2 > 0$ $\Rightarrow f < 0$

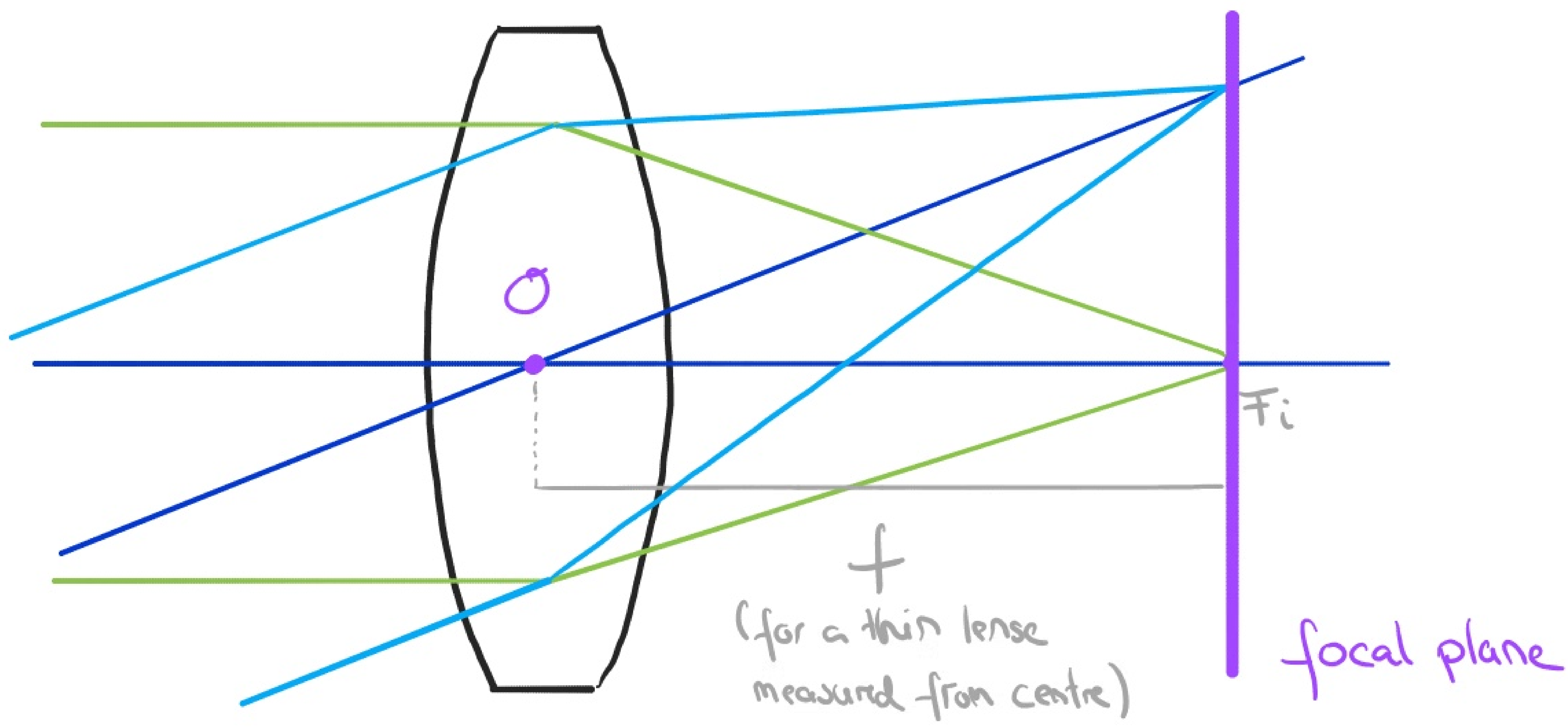
In the above sketches, we have considered on-axis point sources or on-axis plane waves, in which cases it is convenient to draw a ray through the centre of the lens. This ray is perpendicular to both surfaces and, thus, undeviated. We now have to answer the question of what happens, when the point source is located off the optical axis or the plane wave is incident at an angle. To determine how the lens affects the light, we can use the concept of the optical centre of a lens. Any ray that passes through this centre will emerge parallel to its initial direction. We can see this by drawing two parallel planes that are tangent to the interface at the points of incidence. For any ray passing through the optical centre, the lens will act like a plate of glass and only displace it laterally. This displacement is proportional to the thickness of the lens, and thus negligible for a thin lens. The beam appears undeflected and can be drawn as a straight line.



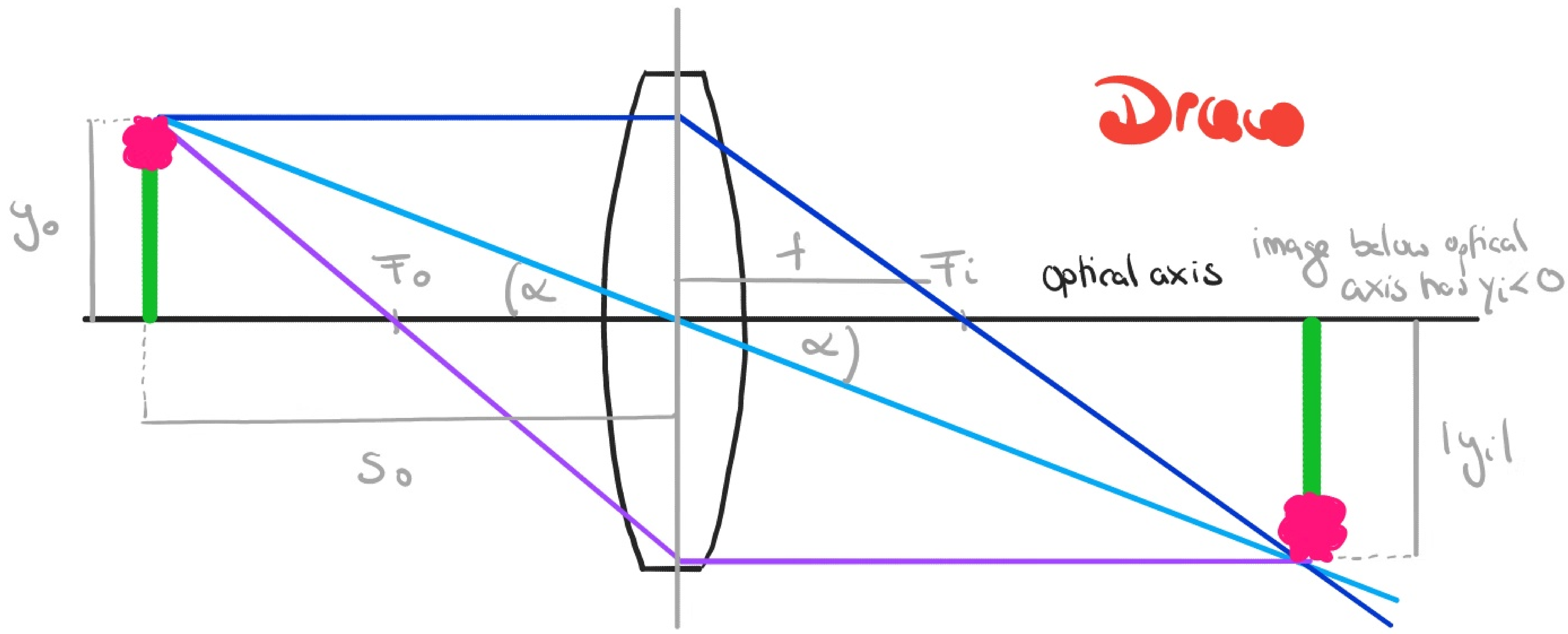

Additionally, we know that in the paraxial approximation, parallel rays incident on a spherical surface are focussed onto a single point on the optical axis. Now imagine additional bundles that hit the interface off axis (see right sketch). Their focal points will lie on a sphere, which for small angles resembles very nearly a plane.

We can now combine these two concepts to determine the behaviour of off-axis rays. Specifically, we define the focal plane of a lens as follows



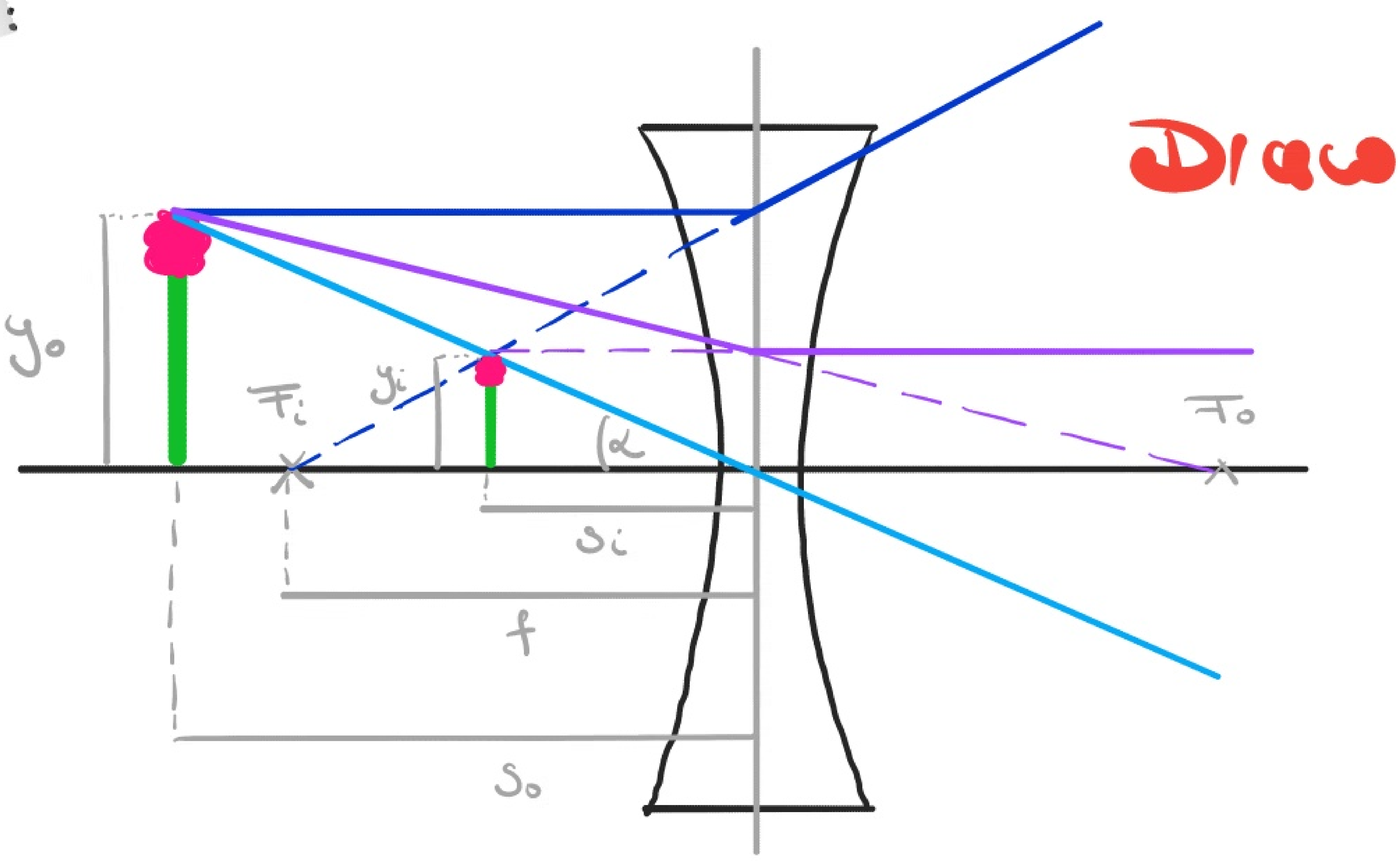

## 2.) Imaging with lenses

The above sketch can be used to understand how lenses can image an object. Each point on the object can be thought of as a point source (e.g. the light reflected diffusely off it). Following their respective rays through an optical system, thus, allows us to build a theory of how images of finite objects propagate. In the paraxial approximation, a planar object will produce also a planar image and we can determine its size, location and orientation using ray diagrams. Any two rays will suffice to image a specific point of the object onto the image and, because we know the position of the focal points, there are three rays that are particular easy to follow:




- The ray going through the optical centre of the lens
- The ray propagating parallel to the optical axis going through the image focal point $F_i$.
- The ray going through the object focus $F_o$, which will be collimated on the other side of the lens.

In the above case, the resulting image is below the optical axis, i.e. inverted.
We can use the same concept to construct the virtual image of a concave lens:



In both cases, we can define the 'transverse magnification' as the ratio of the object's lateral dimension over that of its image. Inspection of the geometry (more precisely the triangles that contain $\alpha$ and are bound by $y_o$ and $s_o$ or $y_i$ and $s_i$) leads to

$$M_T \equiv \frac{y_i}{y_o} = -\frac{s_i}{s_o} .$$

Note that for any real image, $s_o$ and $s_i$ are positive, which suggests that $M_T < 0$ and the image is always inverted. Similarly, for a virtual image, we have $s_i < 0$ and hence $M_T > 0$ so that the image is right-side-up. Finally, as e.g. observed for the concave lens, we can certainly have situations with $|M_T| < 1$. In this case, the object is larger than the image and the term 'magnification' somewhat misleading.

Question: Where does the object have to be positioned with respect to a convex lens to give $|M_T| < 1$, i.e. a smaller image? Where have you encountered this situation before?

Any real object will typically occupy a three-dimensional space and thus have a thickness along the optical axis. The 'longitudinal magnification' can thus be defined as the ratio of the infinitesimal axial length in the region of the image to the corresponding length in the region of the object:

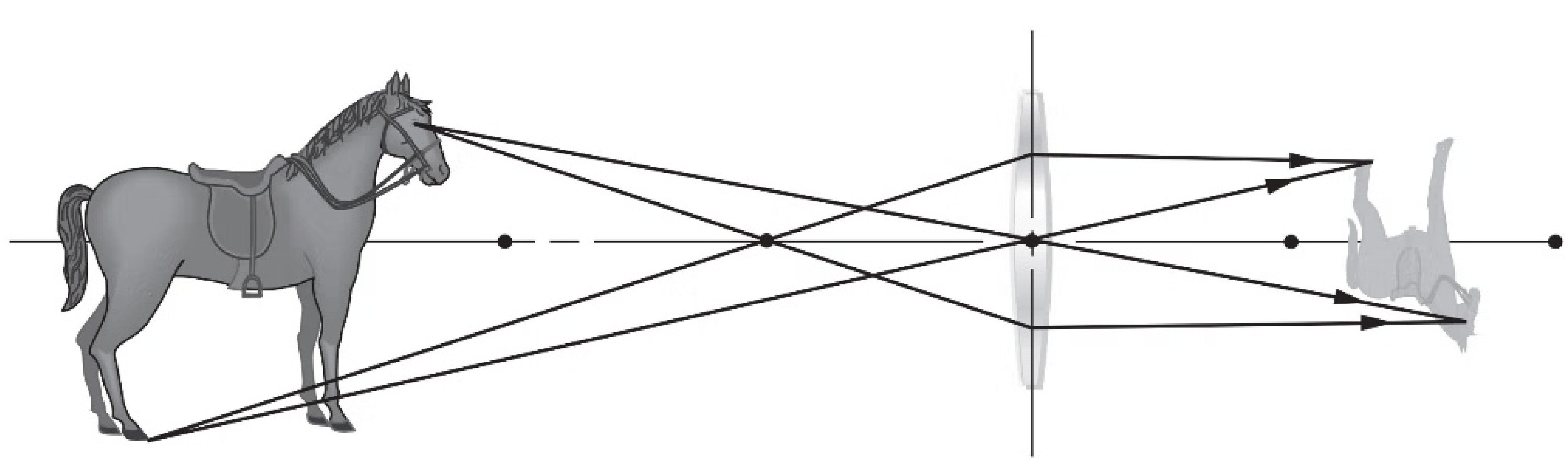
$$m_L \equiv \frac{ds_i}{ds_o}.$$

We can derive a different expression for $m_L$ by differentiating the Gaussian lens formula (see Lecture 5) with respect to $s_o$

$$\frac{d}{ds_o}\Big(\frac{1}{s_o} + \underbrace{\frac{1}{s_i}}_{\text{depends on } s_o} - \underbrace{\frac{1}{f}}_{\text{constant}}\Big) \overset{\text{chain rule}}{=} -\frac{1}{s_o^2} - \frac{1}{s_i^2}\frac{ds_i}{ds_o} = 0,$$

$$\Rightarrow \quad m_L = \frac{ds_i}{ds_o} = -\frac{s_i^2}{s_o^2} = -m_T^2.$$

We observe that the lens affects the transverse and longitudinal magnification differently, causing a distortion in the image. Moreover, we always have $m_L < 0$, implying that if $ds_o$ is positive $ds_i$ has to be negative and vice versa. These two aspects are illustrated in this sketch:



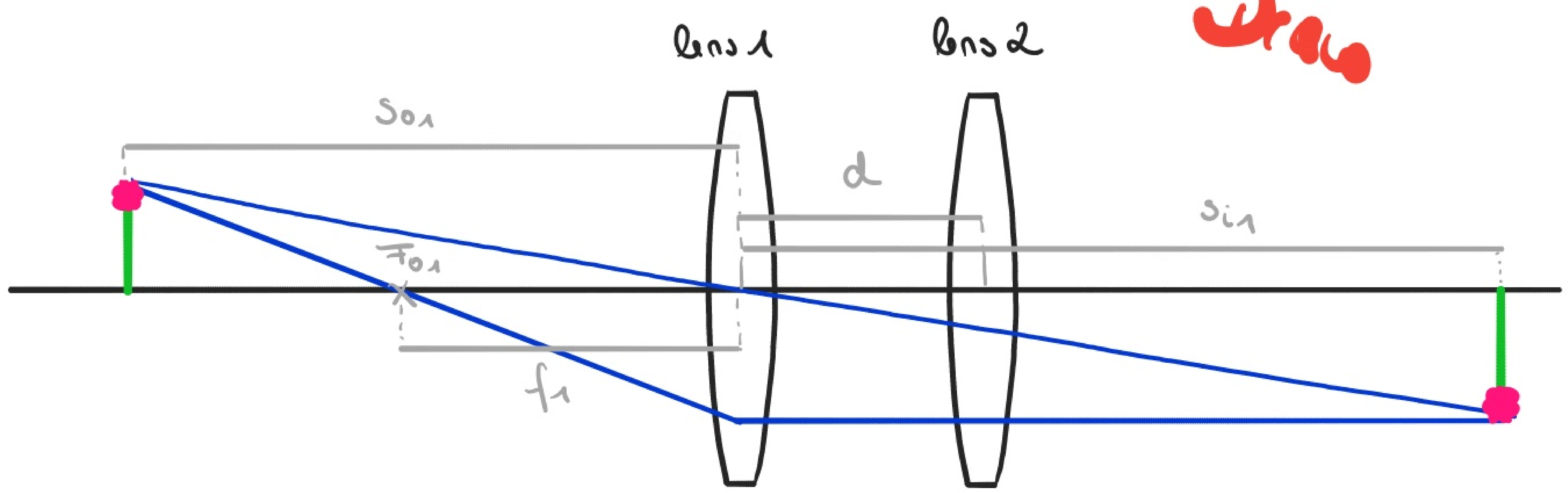
head of object points towards the lens but away in the image, because $m_L < 0$

## 3.) Thin-lens combinations

The strategy for dealing with multi-lens systems is entirely analogous to to our derivation of the thin-lense-formula for two spherical interfaces. We first find the image (real or virtual) of the object created by the first lens. We then use this image as the object for the second lens, finding the second image ... and continue this for all the lenses present.

We will illustrate this concept by analysing the image formed by two thin positive lenses, separated by a distance $d$ that is smaller than either focal length. Constructing the first image, we obtain

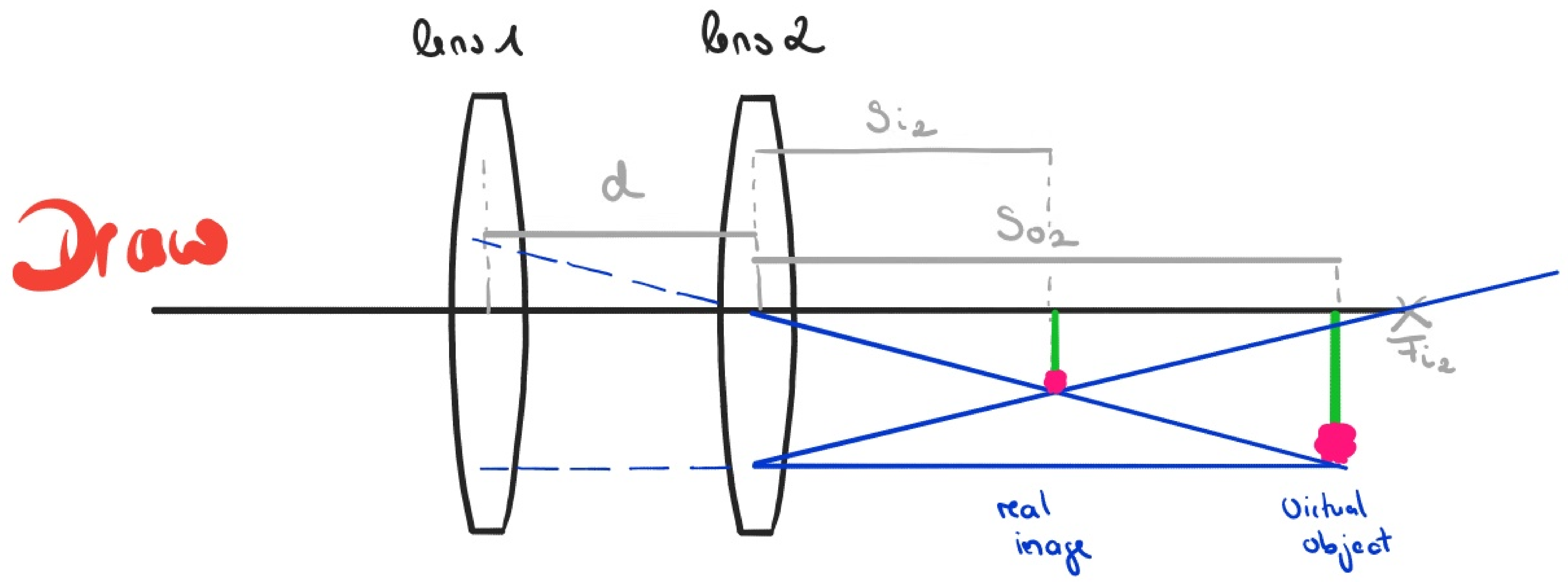
Draw




Using the Gaussian lense formula, we obtain

$$\frac{1}{s_{i1}} = \frac{1}{f_1} - \frac{1}{s_o} \quad \Rightarrow \quad s_{i1} = \frac{s_{o1} f_1}{s_{o1} - f_1} \, . \qquad (*)$$

Constructing the second image leads to




The resulting image is real, minified and inverted. The geometry suggests $S_{o2} = d - S_{i1}$ (which has to be negative, because the virtual image is to the right of the lense) and using Gauss's' lens-formula again, we have

$$\frac{1}{S_{i2}} = \frac{1}{f_2} - \frac{1}{S_{o2}} \Rightarrow S_{i2} = \frac{f_2 S_{o2}}{S_{o2} - f_2} = \frac{f_2 (d - S_{i1})}{d - S_{i1} - f_2} . \quad (**)$$

Combining both expressions, we arrive at a single expression

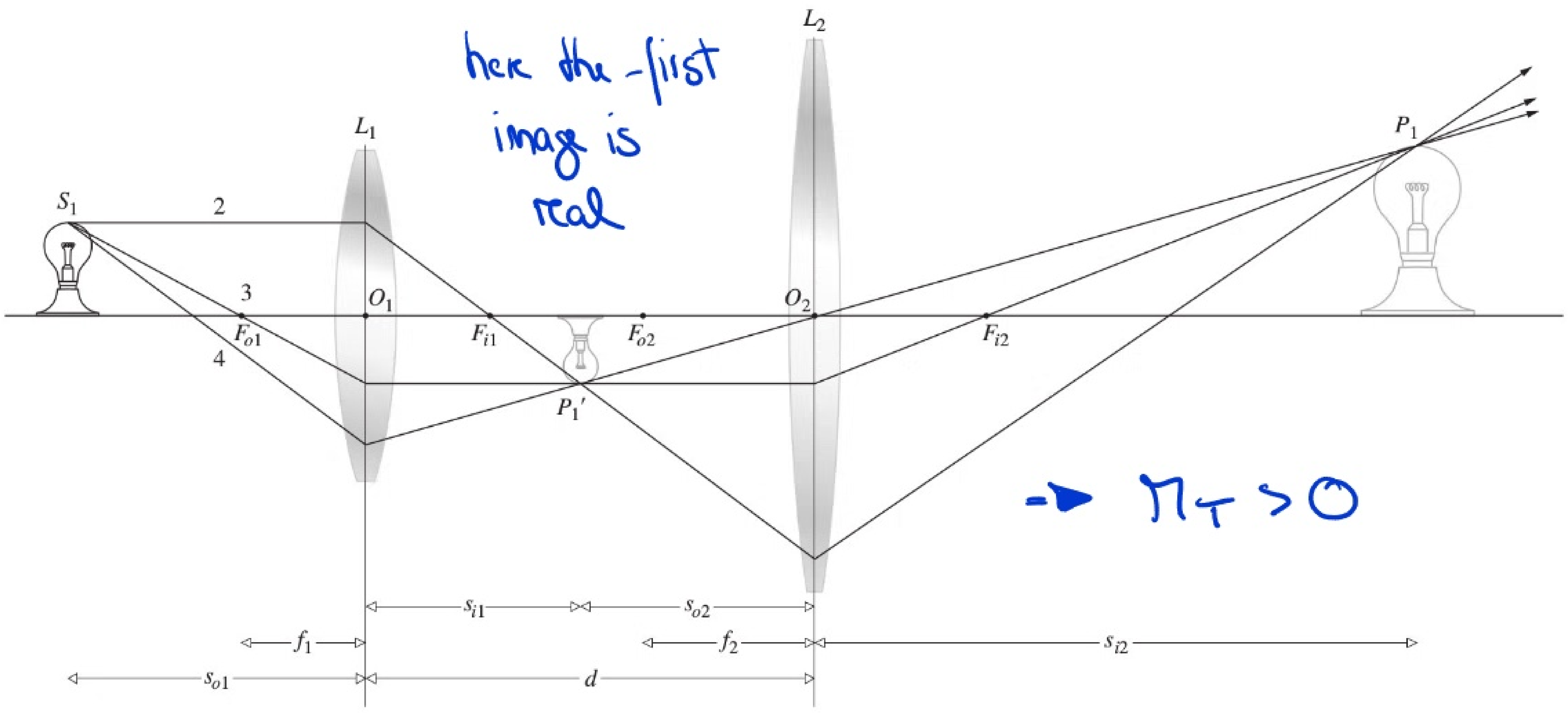
$$S_{i2} = \frac{f_2 d - f_2 f_1 S_{o1} / (S_{o1} - f_1)}{d - f_2 - f_1 S_{o1} / (S_{o1} - f_1)} ,$$

where $S_{o1}$ and $S_{i2}$ are the object and images of the compound lens system. The transverse magnification of the two lenses is given by

$$M_T = M_{T1} \cdot M_{T2} = \frac{S_{i1}}{S_{o1}} \cdot \frac{S_{i2}}{S_{o2}} .$$

In the above example, $S_{i2} < 0$ which gives $M_T < 0$ as observed.

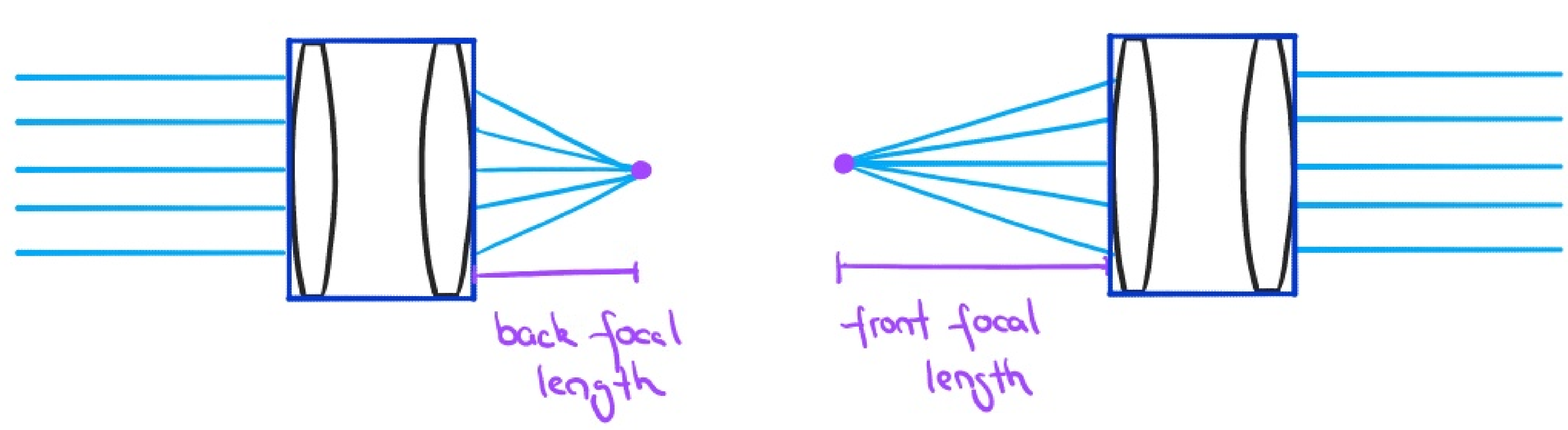
Not that we can follow the same procedure to construct the image emerging from two positive lenses separated by a distance $d$, greater than the sum of their focal lengths. The ray diagram would look as follows




$\rightarrow M_T > 0$

The distance from the last interface of an optical system to the second focal point is known as the back focal length, while the distance from the first focal point to the first interface is the front local length:

Draw




In general, the two focal lengths are different (they are only equal in the limit $d \rightarrow 0$) and for a two lense system, we can determine

bfl: $\quad s_{o1} \rightarrow \infty$ so $\text{bfl} = s_{i2} = \dfrac{f_2(d-f_1)}{d-(f_1+f_2)}$, ffl : $s_{i_2} \to \infty$ so $\text{ffl} = s_{o_1} = \dfrac{f_1(d-f_2)}{d-(f_1+f_2)}$.

subscripts are switched !!

In the limit $d \to 0$, where the lenses are in contact, both expressions give

$$\text{bfl} = \text{ffl} \equiv f = \frac{f_1 f_2}{f_1+f_2} \quad \text{or} \quad \underline{\underline{\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}}},$$

where $f$ is referred to as the effective focal length. This expression can be generalised to a multi-lens system (provided that the total thickness of the compound lens is negligible)

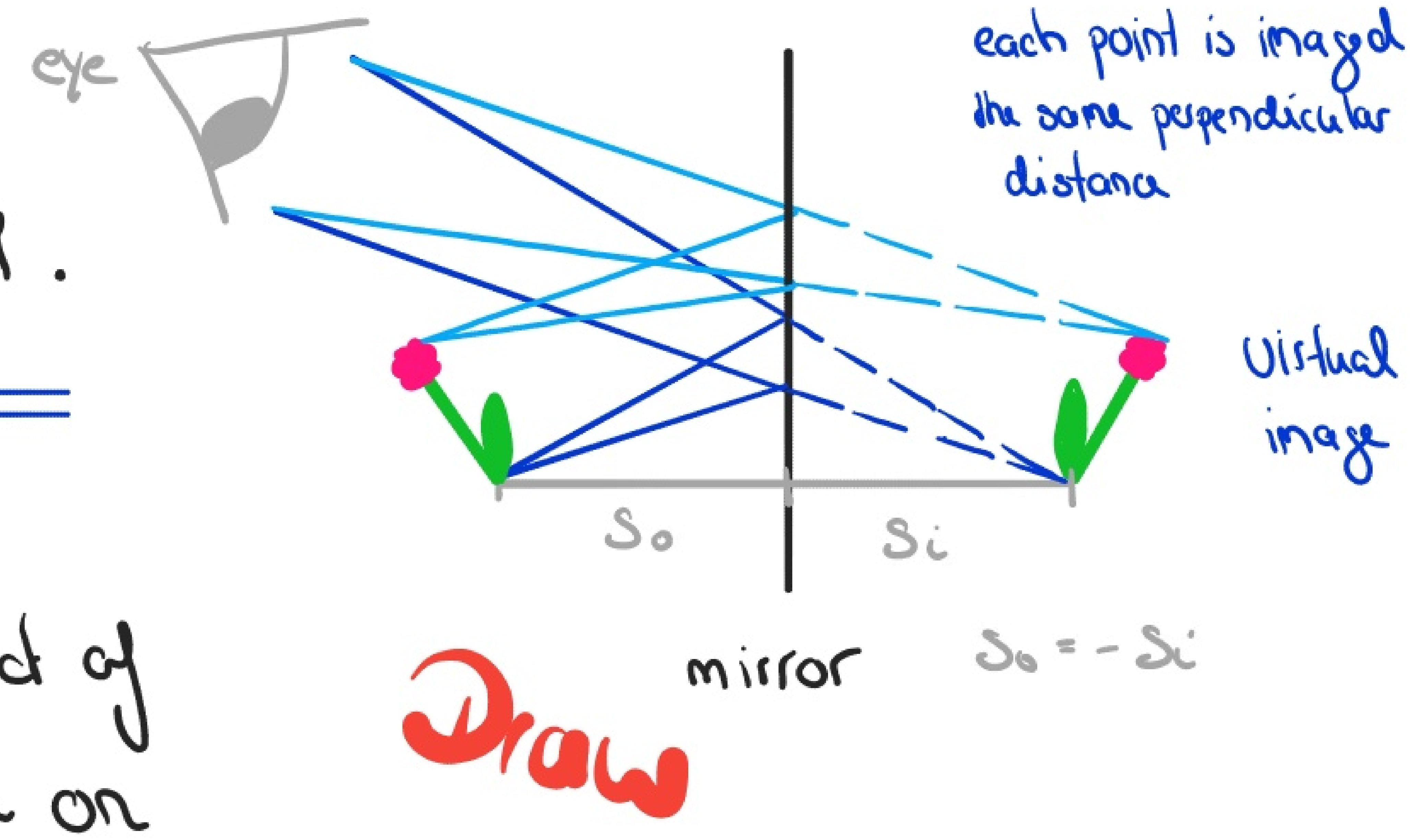
$$\underline{\underline{\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \dots + \frac{1}{f_N}}}.$$

## 4.) Imaging with mirrors

Mirrors are typically created by applying a reflective coating to a suitable base. The coating material is usually a metal, whose reflective properties are well described within the Lorentz model in the limit $\omega_0 \to 0$ (electrons can be freely accelerated in a metal so there is no restoring force). In this case, the refractive index reads $n^2 = 1 - \omega_p^2/\omega^2$, where $\omega_p^2 = Nq_e^2/\varepsilon_0 m_e$ is the plasma frequency (see Problem 2, PS ~~X~~1). $\omega_p$ is typically in the UV, so that for optical frequencies with $\omega_p > \omega$, $n$ becomes purely imaginary. This implies strong absorption but also strong reflection.

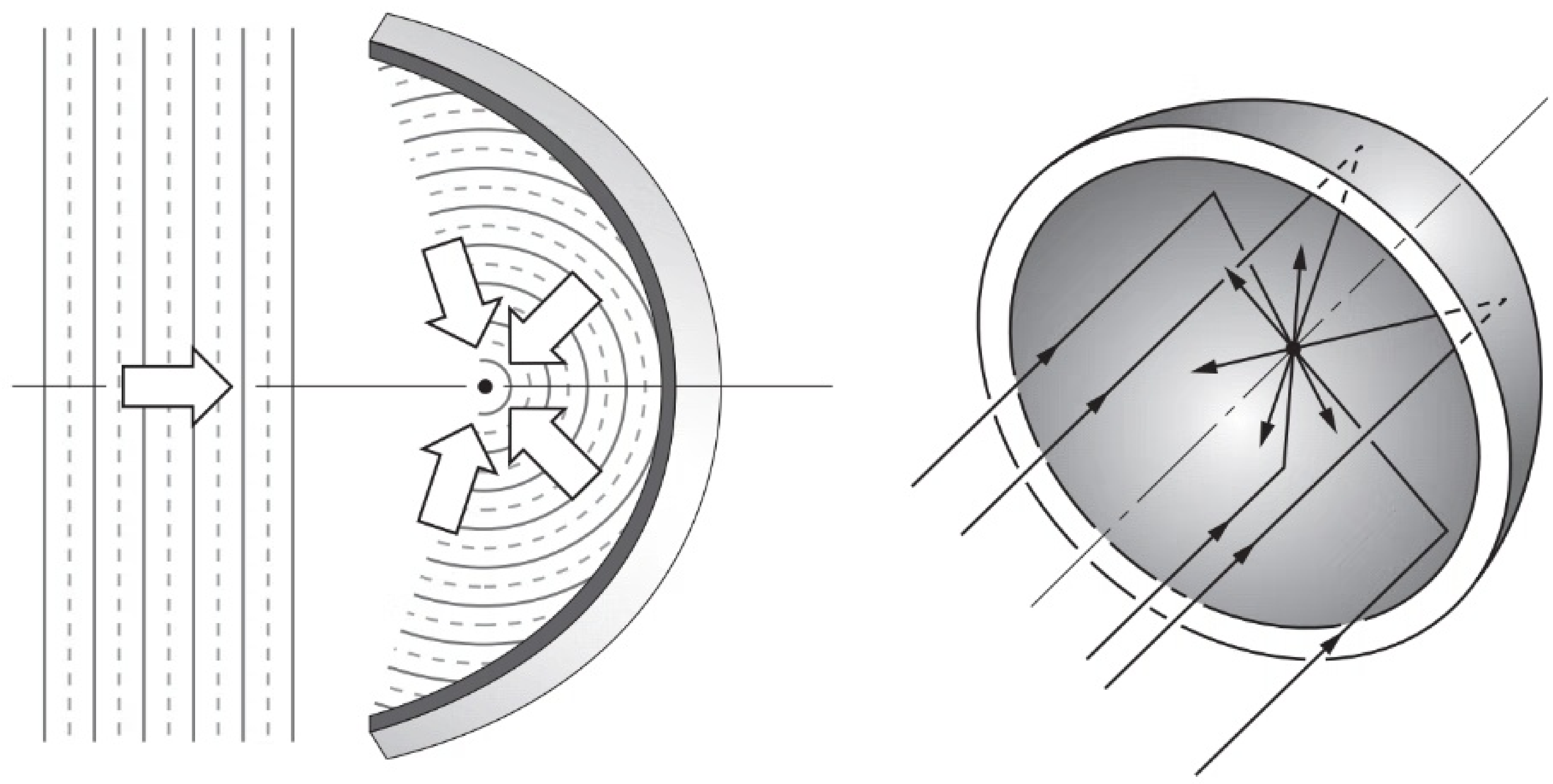
The most common mirrors are planar and we can translate the formalism introduced for lenses directly to construct image formation on a mirror. Using the law of reflection, we can construct a virtual image of the object on the other side of the mirror. Defining $s_i$ to be a negative quantity, if S or P are positioned to the right of the reflective surface, we have

$$n_T = -\frac{s_i}{s_o} = \frac{|s_i|}{s_o} = 1 .$$

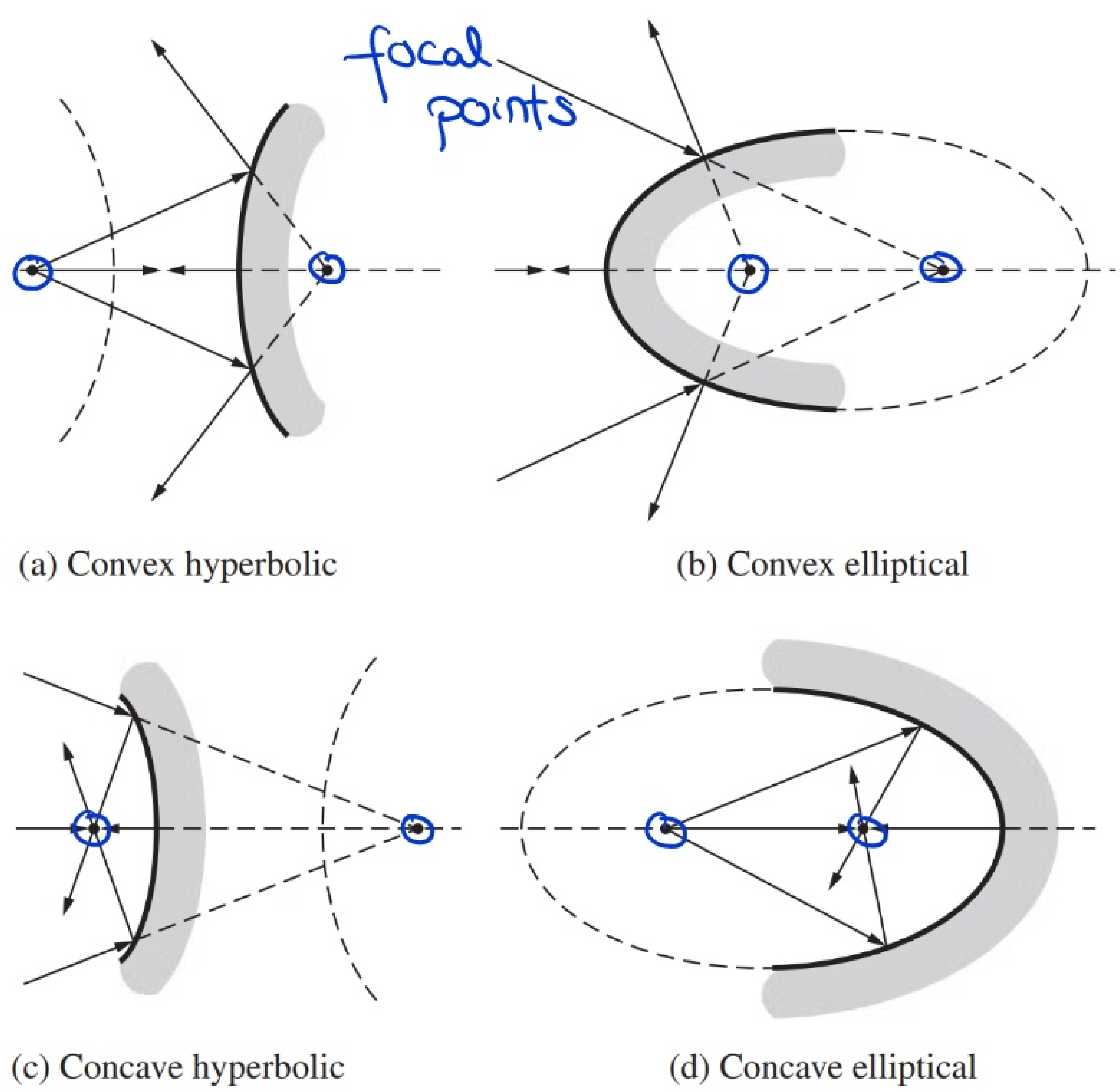



A key difference to the impact of a lens is that the reflection on the mirror changes the handedness of the object. If you look e.g. at your left hand in the mirror, it will appear like your right hand. Two reflections will however restore the original handedness of the object.

As illustrated in the sketch above, planar mirrors cause the rays to diverge and cannot be used to collimate / focus light. In principle, aspherical surfaces are required to perfectly collimate / focus (very much in the same way as we encountered for lenses) and Fermat's principle can again be invoked to determine the shape that is necessary to do so. The resulting mirror shape is parabolic (because the two media involved are identical $n_1 = n_2$) and shown on the next page.



Additionally, elliptic and hyperbolic mirrors are used for their respective reflective properties. We again distinguish concave and convex mirrors depending on whether the centre of curvature occurs on the side of the reflecting surface or the opposite side as illustrated below




(a) Convex hyperbolic

(b) Convex elliptical

(c) Concave hyperbolic

(d) Concave elliptical

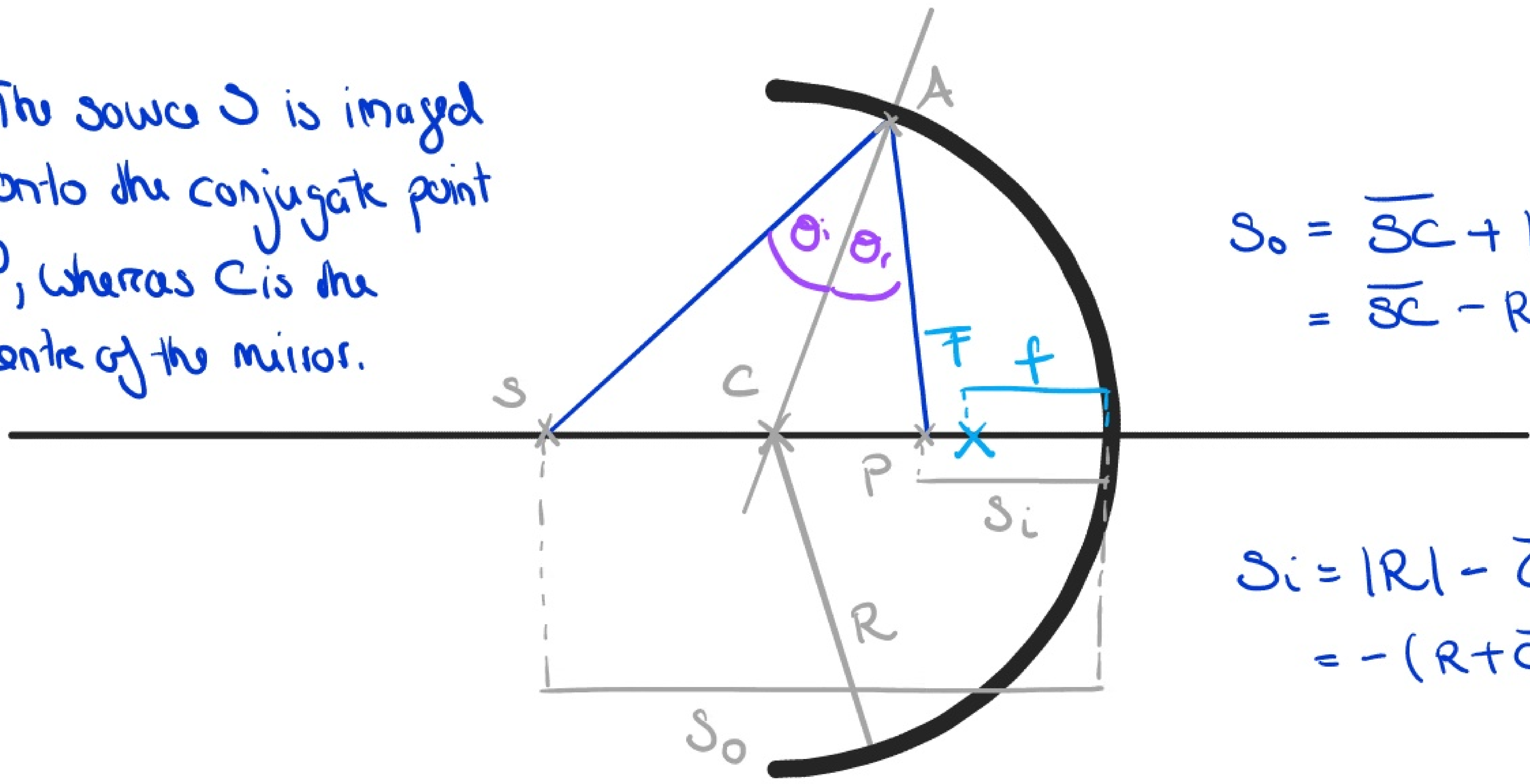
Elliptical mirrors focus light from one focal point onto the other.

The primary mirror of the Hubble Space Telescope is a concave hyperbolic !!

As for lenses, manufacturing perfectly aspherical surfaces is very difficult, so

So most standard mirrors are spherical. In the paraxial approximation, they can be used for imaging in much the same way as lenses. Consider the following geometry with a spherical mirror with curvature radius $|R|$. Note that as a convention, we choose $R$ to be negative for a concave mirror, while $s_i$ and $s_o$ are defined to be positive.

The source S is imaged onto the conjugate point P, whereas C is the centre of the mirror.




$$s_o = \overline{SC} + |R| = \overline{SC} - R$$

$$s_i = |R| - \overline{CP} = -(R + \overline{CP})$$

In the paraxial approximation, we have $s_o \approx \overline{SA}$ and $s_i \approx \overline{PA}$. Also since $\theta_i = \theta_r$, we can relate the sides of the blue triangle to the sections between S and P along the optical axis

$$\frac{\overline{SC}}{\overline{SA}} = \frac{\overline{CP}}{\overline{PA}} \Rightarrow \frac{s_o + R}{s_o} = -\frac{s_i + R}{s_i} .$$

Rearranging gives the mirror formula

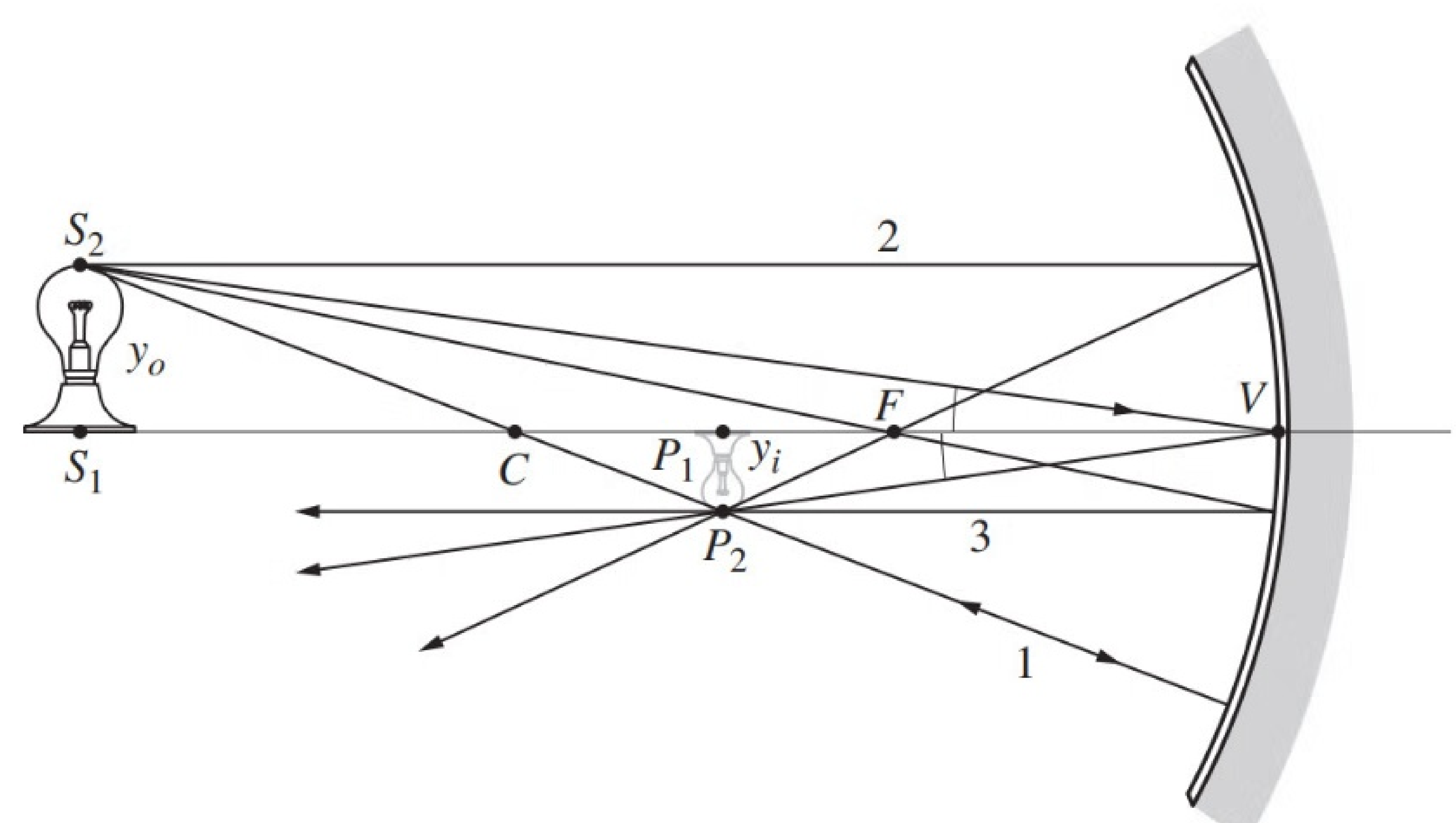
$$\frac{1}{s_o} + \frac{1}{s_i} = -\frac{2}{R} .$$

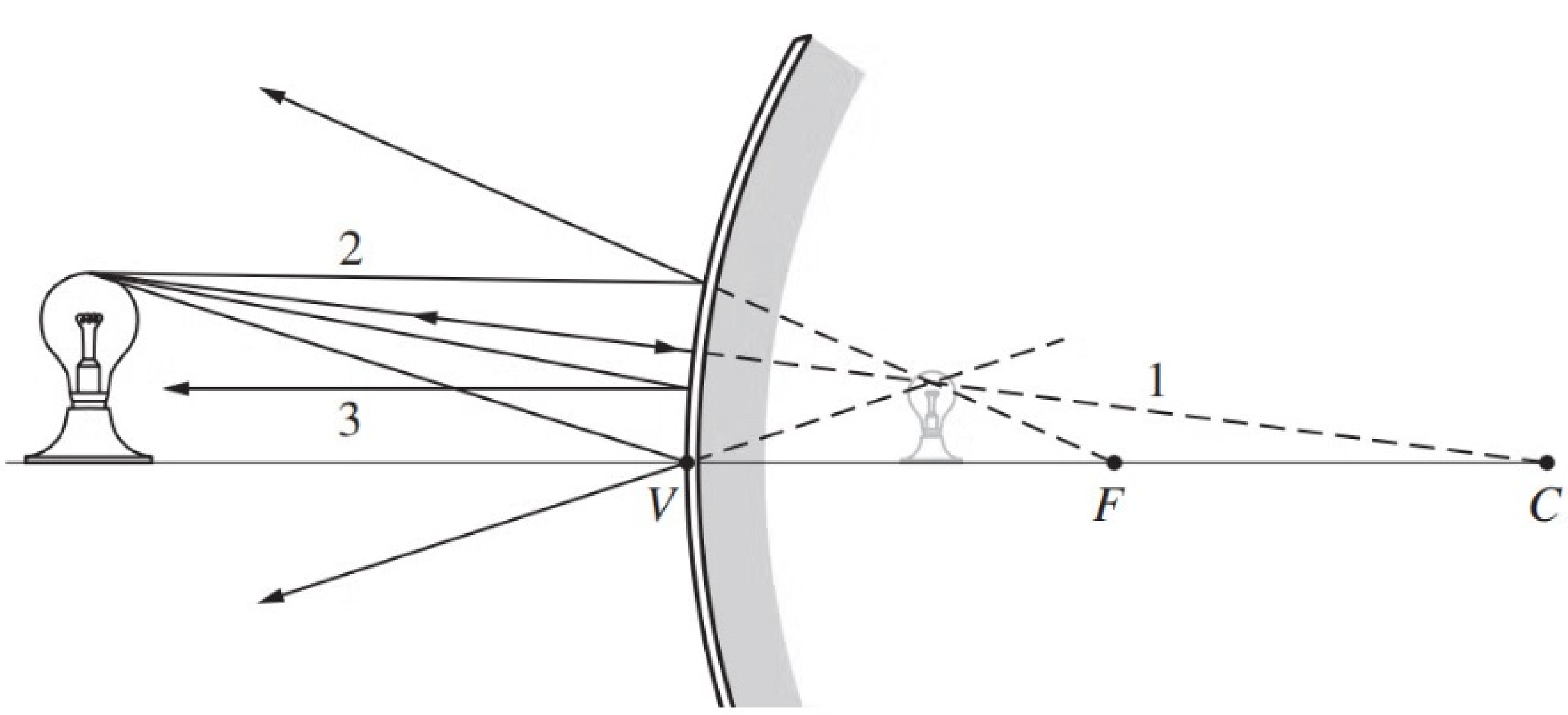
Defining the focal lengths as before ( $s_{i,o} \to \infty$), we obtain

$$f_o = f_i \equiv f = -\frac{R}{2} \quad \text{or} \quad \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} .$$

We observe that $f$ will be positive for concave mirrors with $R<0$ and negative for concex mirrors. Using the concept of ray diagrams, we can readily construct the images created by mirrors, i.e. rays going through the focus will be collimated after hitting the mirror, while rays going through the centre of the mirror hit the surface perpendicular and will bounce back the same way they entered.

Real, inverted image that is minified if $S_o > 2f$.




Virtual image, that is right-side-up and minified.

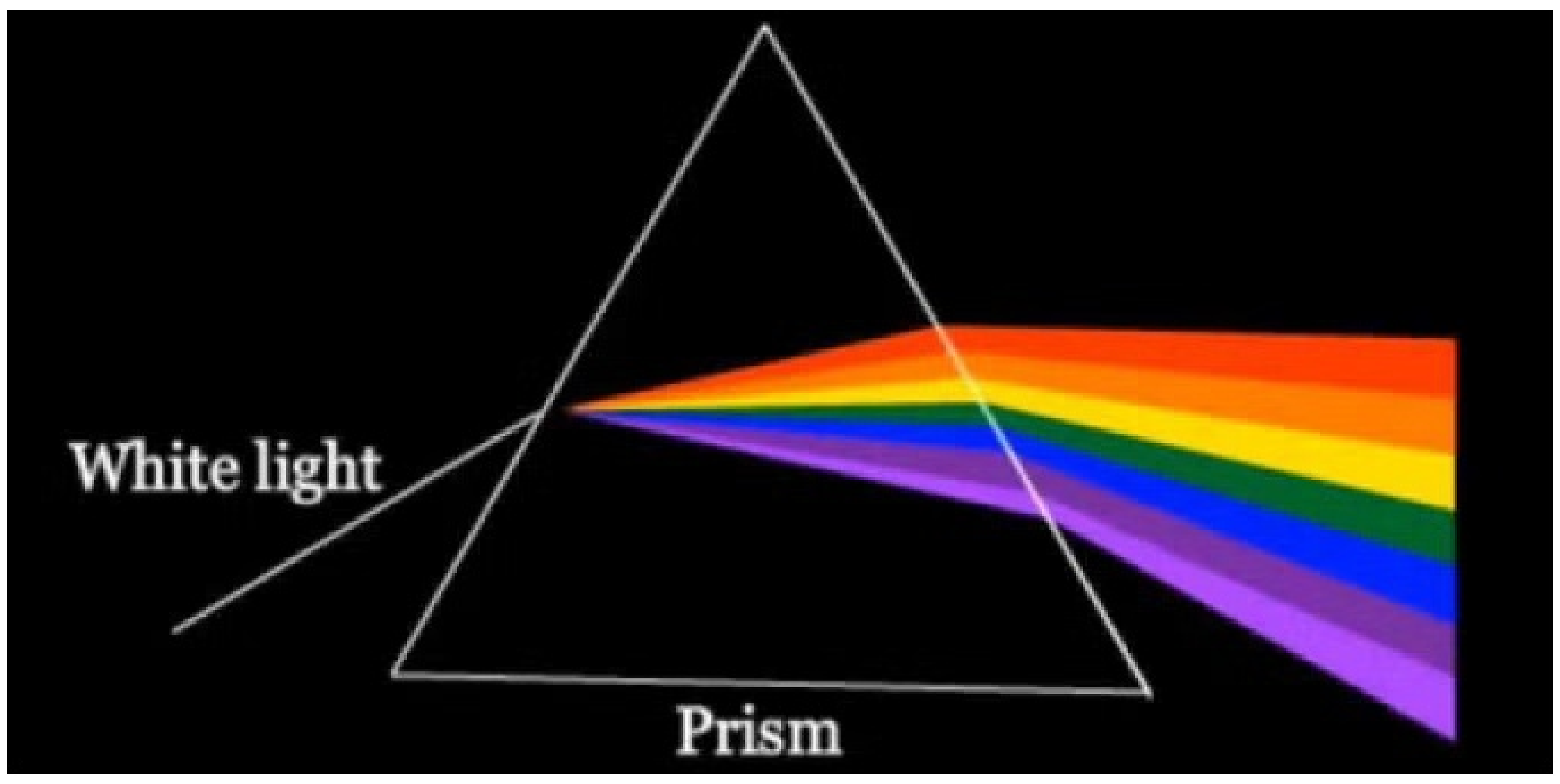
# PHYS 434 - Lecture 7

# Prisms, Optical systems, Grav. Lensing

## 1.) Prisms

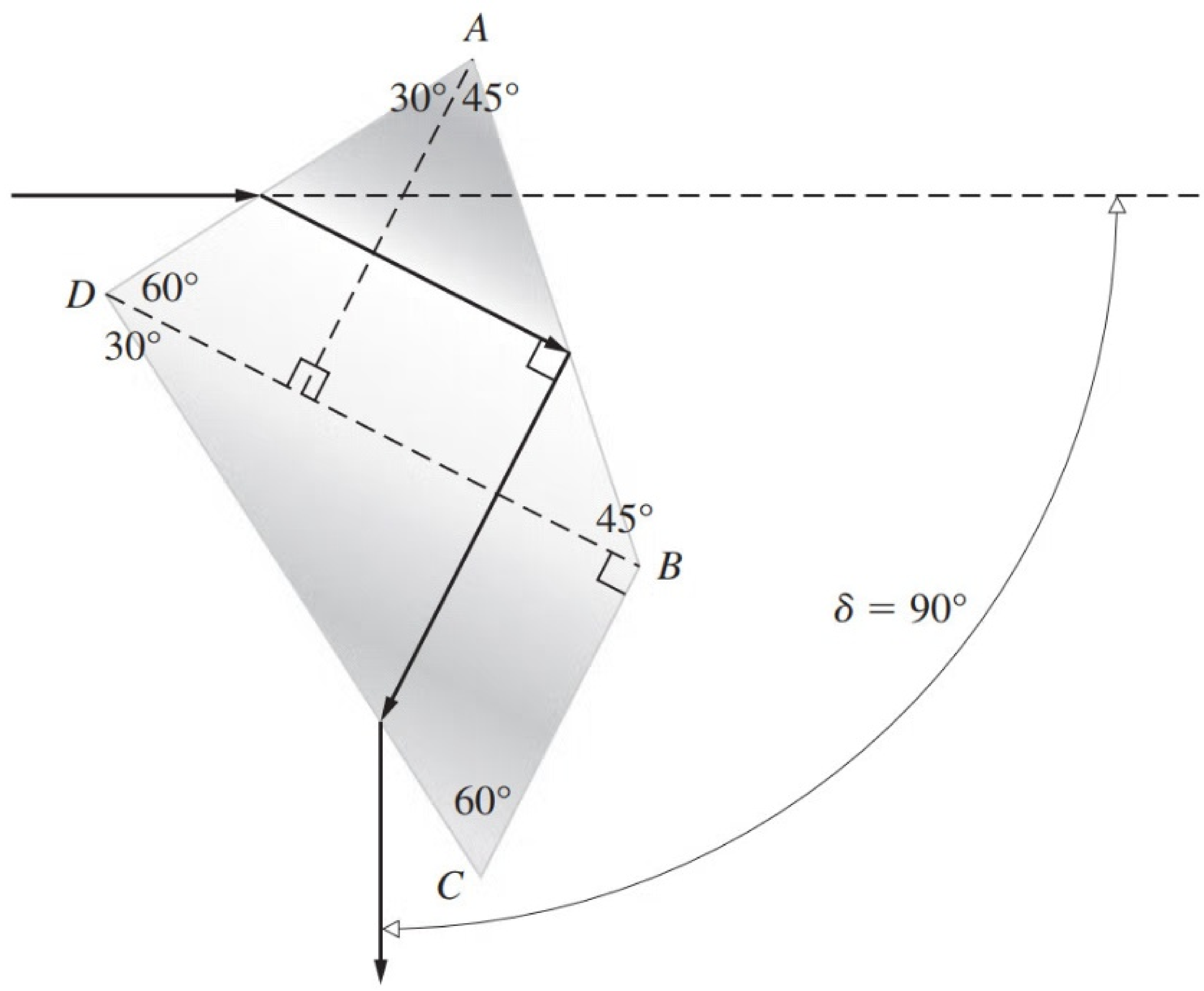
These optical devices are primarily used for two main reasons: (i) they are dispersive and can be e.g. used to analyse spectral properties of light, (ii) prisms can change the direction of a beam or the direction of an image and thus useful to fold systems into confined spaces.

We already encountered the dispersing prism in Lecture 3, as an example of applying the law of refraction. Specifically, we derived an expression for the deflection angle $\delta$ as a function of the incidence angle $i$, the prism angle $\alpha$ and the refractive index $n$. As $n$ itself is a function of frequency / wavelength, $\delta$ will vary with wave length. More precisely, $\delta$ increases as $n$ increases, which suggests that since $dn/d\omega > 0$ high frequencies are deflected more than low frequencies, resulting in



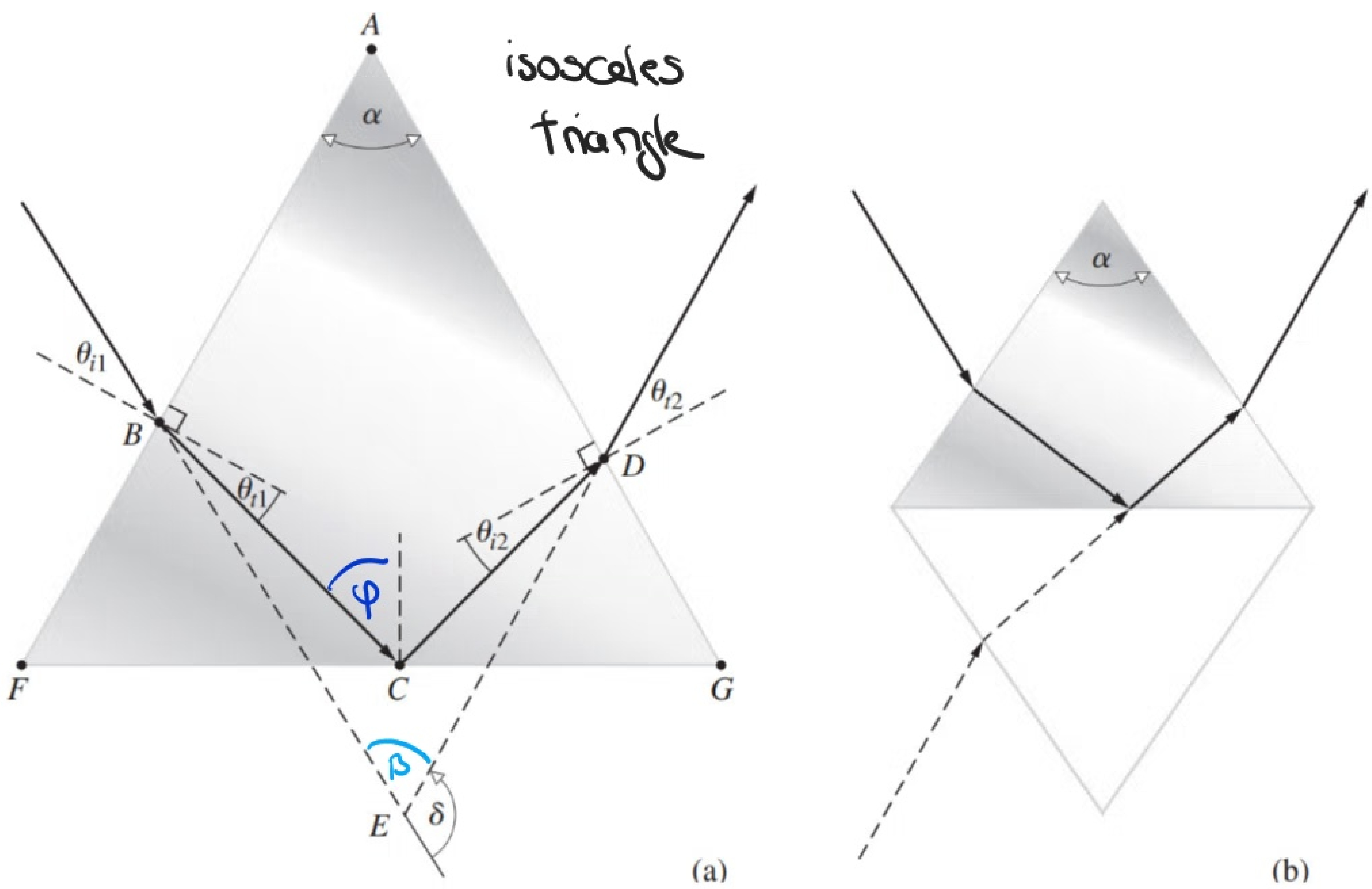

(drawn very exaggerated)

Using the same concept but a prism of a different shape, it is possible to employ refraction, reflection and dispersion in a way that allows us to extract specific wavelengths at a fixed angle. Such constant-deviation prisms are important for spectroscopy and the path through a so-called Pellin-Broca prism is illustrated below




A specific wavelength $\lambda$ is traversing the prisms in a symmetric manner and leaves the glass at $\delta = 90°$. By slightly rotating the prism a different $\lambda$ is selected to emerge at $90°$.

In a lot of cases, however, dispersion is not desired. This can be achieved by introducing the beam in such a way that at least one total internal reflection takes place to change the beam / image direction. The geometry is given on the next page. For internal reflection to take place, we require $\varphi > \theta_c$ (remember $\sin\theta_c = n_t / n_i$; for glass $\theta_c \approx 42°$) at point C. The deflection angle is now given by $\delta = 180° - \beta$ and by inspecting the geometry, we can relate $\beta$ to $\theta_{i1}, \theta_{t2}$ and $\alpha$. Moreover, since the prism is



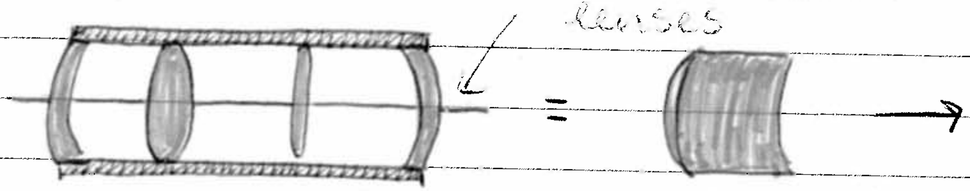

isosceles, we also have $\theta_{t1} = \theta_{i2}$ and thus $\theta_{i1} = \theta_{t2}$ in which case we arrive at $\delta = 2\theta_{i1} + \alpha$. This expression is independent of $\lambda$ and $n$ and hence reflection will take place without any preference for colour. Such a prism is said to be achromatic. This becomes obvious if we extend the prism as in the left sketch: The prism is equivalent to a thick plate and the image of the incident ray emerges parallel to itself independent of its wavelength.

# PHYS 434 Lecture 8 : Matrix methods

So far, we have been studying only thin lenses in the paraxial approximation. Already, the algebra can become quite cumbersome when working with multi-element optical systems (e.g. the compound microscope in lecture 7).

More sophisticated optical analysis can be done by ray-tracing techniques. Notably, these can be formulated into matrix multiplication in the paraxial approximation. We will show that

same optical axis for all lenses



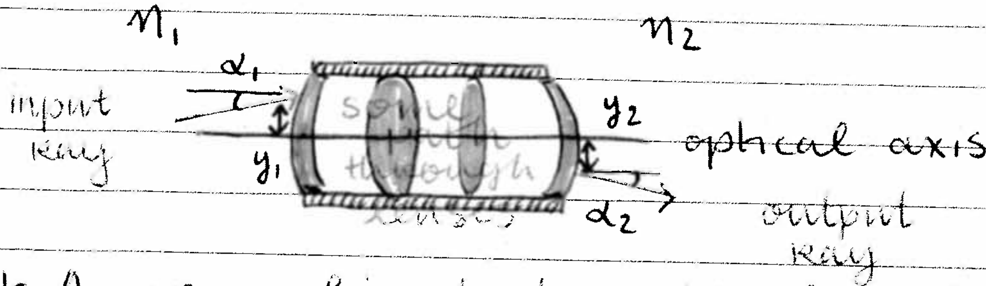
compound lens

equivalent thick lens (different for each compound lens system)

can be represented by a 2x2 matrix (in the paraxial approximation, for meridional rays)

non-negligible distance between two spherical lens ~~syst~~ surfaces

Why ~~can~~ did someone come up with the idea that we can represent optical elements as matrices?




A complicated optical element is a relationship between each input ray and associated output ray

$\{n_2, \alpha_2, y_2\}$ = some function of $\{n_1, \alpha_1, y_1\}$

↑ this mathematical object describes our compound lens completely

If relation is linear, the object will be a matrix →

To figure out the details, we will need to examine how rays propagate through spherical interfaces from this "ray-tracing" perspective.

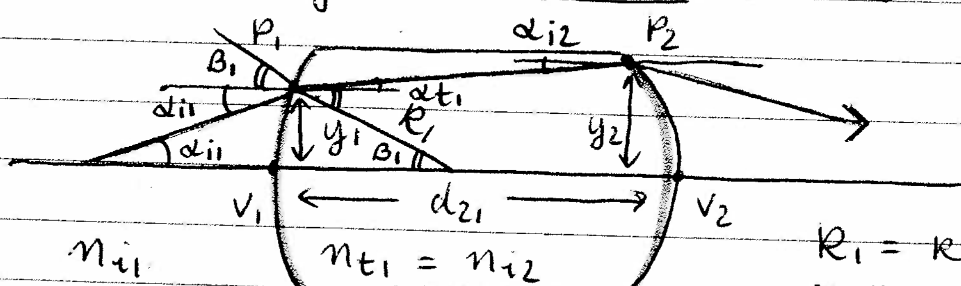
## Ray tracing & Matrix methods

Modern optical design is done by mathematically propagating rays through an optical system and applying

Snell's law at each interface. We can ~~do this analytically~~ understand this process by analytical techniques if we restrict ourselves to

* the paraxial approximation
  $n_1 \sin\theta_1 = n_2 \sin\theta_2 \Rightarrow n_1\theta_1 = n_2\theta_2$
* meridional rays
  ( rays in the same plane as the optical system ( rays do not come out of the paper)

**Example**: A paraxial, meridional ray traversing a thick lens




$R_1$ = radius of curvature of 1st spherical interface

Now, consider the refraction of the ray at point $P_1$. The incidence ray is completely specified by the incidence angle $\alpha_{i1}$ and the incidence position $y_{i1} = y_1$

Snell's law $\quad n_{i1} \sin(\alpha_{i1} + \beta_1) = n_2 \sin(\alpha_{t1} + \beta_1)$

$$\Rightarrow n_{i1}\left(\alpha_{i1} + \frac{y_1}{R_1}\right) = n_{t1}\left(\alpha_{t1} + \frac{y_1}{R_1}\right)$$

$\sin(\theta) \approx \theta$
$\beta_1 \approx y_1 / R_1$ by paraxial approximation

$$\Rightarrow \quad n_{t1}\,\alpha_{t1} = n_{i1}\,\alpha_{i1} - \boxed{\frac{(n_{t1}-n_{i1})}{R_1}}\, y_1$$

$$\Rightarrow \quad \boxed{n_{t1}\,\alpha_{t1} = n_{i1}\,\alpha_{i1} - \mathcal{D}_1\, y_{i1}}$$

refraction equation

"dioptric power" of the 1st surface

and $\boxed{y_{t1} = y_{i1} = y_1}$

no position change immediately at the surface

The next step to consider is the straight-line propagation through constant-index regions. Consider how $y_{i2} = y_2$ is related to $y_{t1}$ and $\alpha_{t1}$: $\leftarrow = \alpha_{i2}$

$$\boxed{y_{i2} = y_{t1} + d_{21}\,\alpha_{t1}}$$

transfer equation

position after transmission at point $P_1$ $(y_{t1} = y_1)$

in the paraxial approximation

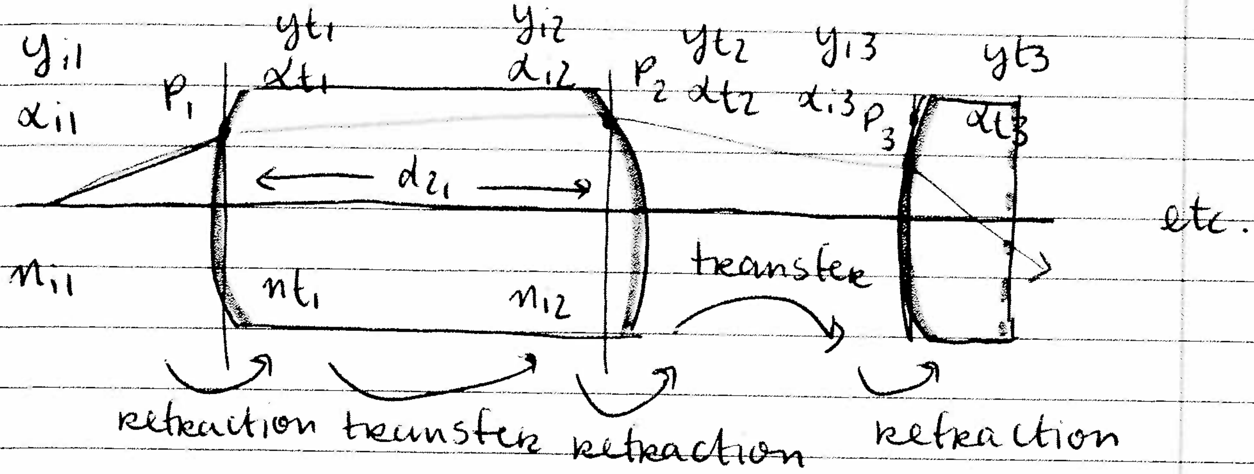
$d_{21} \approx V_1V_2$ or

$d_{21} \approx$ thickness of lens

Note the process by which we can now trace a ray through an arbitrary system:

$$\underbrace{\begin{matrix}\alpha_{i1}\\ y_{i1}\end{matrix} \xrightarrow{\text{refraction}}}_{P_1} \begin{matrix}\alpha_{t1}\\ y_{t1}\end{matrix} \xrightarrow[\;\longleftarrow d_{21} \longrightarrow\;]{\text{transfer}} \begin{matrix}\alpha_{i2}\\ y_{i2}\end{matrix} \underbrace{\xrightarrow{\text{refraction}}}_{P_2} \ \text{etc.}$$

Or, drawn on top of a lens diagram:




Both transfer and refraction involve a linear relationship between the new $\alpha, y$ and the old $\alpha, y$ $\Rightarrow$ can be represented by a matrix!

## Matrix methods strategy

At each point in the ray tracing, we will represent the ray by the vector $\begin{pmatrix} n\alpha \\ y \end{pmatrix}$

Refraction: at the $m^{th}$ interface.

$$\underbrace{\begin{pmatrix} n_{tm}\,\alpha_{tm} \\ y_{tm} \end{pmatrix}}_{\text{ray just after the } m^{th} \text{ interface}} = \underbrace{\begin{pmatrix} 1 & -\mathcal{D}_m \\ 0 & 1 \end{pmatrix}}_{R_m} \underbrace{\begin{pmatrix} n_{im}\,\alpha_{im} \\ y_{im} \end{pmatrix}}_{\text{ray just before the } m^{th} \text{ interface}}$$

dioptric power of the $m^{th}$ interface

$$\mathcal{D}_m = \frac{n_{tm} - n_{im}}{R_m}$$

**Transfer**: For free propagation between interface $p$ and interface $q = p+1$ in material of index $n_{tp} = n_{iq}$

$$\begin{pmatrix} n_{iq}\,\alpha_{iq} \\ y_{iq} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{d_{qp}}{n_{tp}} & 1 \end{pmatrix}}_{\mathcal{T}_{qp}} \begin{pmatrix} n_{tp}\,\alpha_{tp} \\ y_{tp} \end{pmatrix} \leftarrow \text{ray after refraction at } p$$

ray before refraction at $q$

distance between interfaces $p$ and $q$ divided by the index of refraction in that region

Then, we can propagate rays through a system of lenses by matrix multiplication. Using the shorthand

$$r_{ab} = \begin{pmatrix} n_{ab}\,\alpha_{ab} \\ y_{ab} \end{pmatrix} \quad \text{for each ray}$$

$$r_{t1} = \mathcal{R}_1\, r_{i1}$$
$$r_{i2} = \mathcal{T}_{21}\, r_{t1} = \mathcal{T}_{21}\,\mathcal{R}_1\, r_{i1}$$
$$r_{t2} = \underbrace{\mathcal{R}_2\,\mathcal{T}_{21}\,\mathcal{R}_1}\, r_{i1}$$

a matrix that describes the effect of a lens
$\Rightarrow$ "system matrix" for that lens

If we consider a lens of thickness $d_\ell$, index $n_\ell$, in air, with radii of curvature $R_1$ and $R_2$, it is straight forward to calculate its system matrix $\mathcal{A}$

$$A = R_2 \, \mathcal{T}_{21} \, R_1 = \begin{bmatrix} 1 - \dfrac{\mathcal{D}_2 d_l}{n_l} & -\mathcal{D}_2 - \mathcal{D}_1 + \mathcal{D}_1 \mathcal{D}_2 \dfrac{d_l}{n_l} \\ d_l / n_l & 1 - \dfrac{\mathcal{D}_1 d_l}{n_l} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Note that if you take the limit $d_l \to 0$, you find the system matrix for a <u>thin lens</u>

$$A_{\substack{\text{thin} \\ \text{lens}}} = \begin{pmatrix} 1 & -\mathcal{D}_1 - \mathcal{D}_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/f \\ 0 & 1 \end{pmatrix}$$

as $\frac{1}{f} = \mathcal{D}_1 + \mathcal{D}_2$ for a thin lens.

You can then use lens matrices, along with transfer matrices to describe propagation between the lenses, to build up the matrix describing multi-lens systems.

This matrix tells us:

* everything about the lens in the system
* and that any complicated optical system can, in the paraxial approximation, be described by an effective <u>thick</u> lens with the same system matrix.

# Characterizing a (thick) lens system

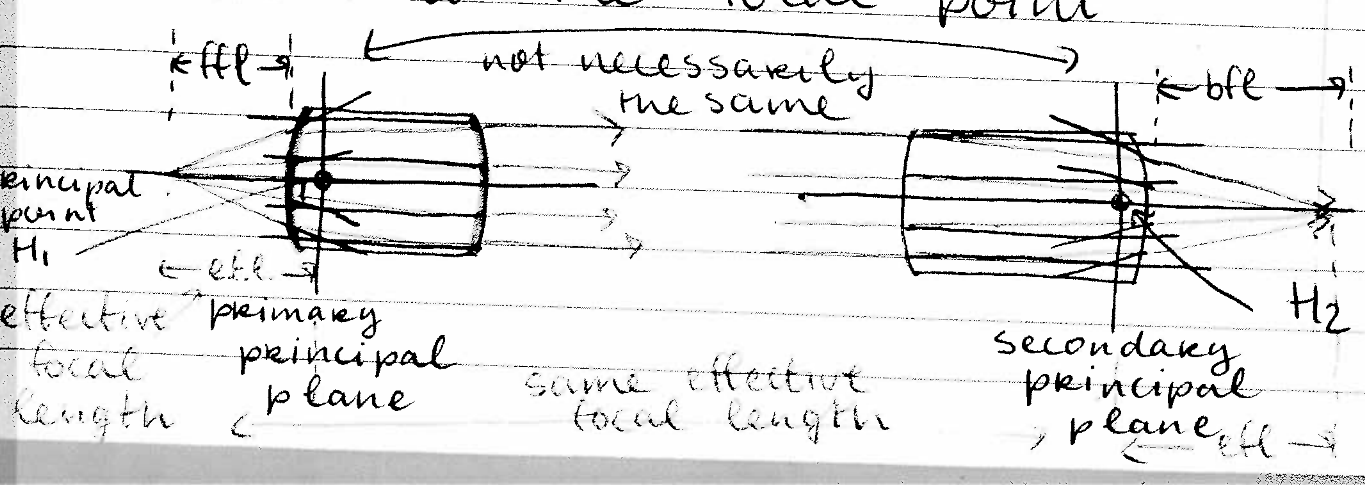
A thick lens (or compound lens) does not necessarily have the same front focal length and back focal length. It is usually characterized by:

* two "principal planes" – the locations where collimated and focusing rays would be extrapolated to intersect
* an effective focal length – equal to the distance between the front and back focus and its respective principal plane.

Amazingly, if you measure the object and image distances from the respective principal planes, then the lensmaker's equation holds.

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f_{eff}}$$

Principal planes: The plane (approximately) where incident and refracted rays would be extrapolated to intersect for a source at the focal point



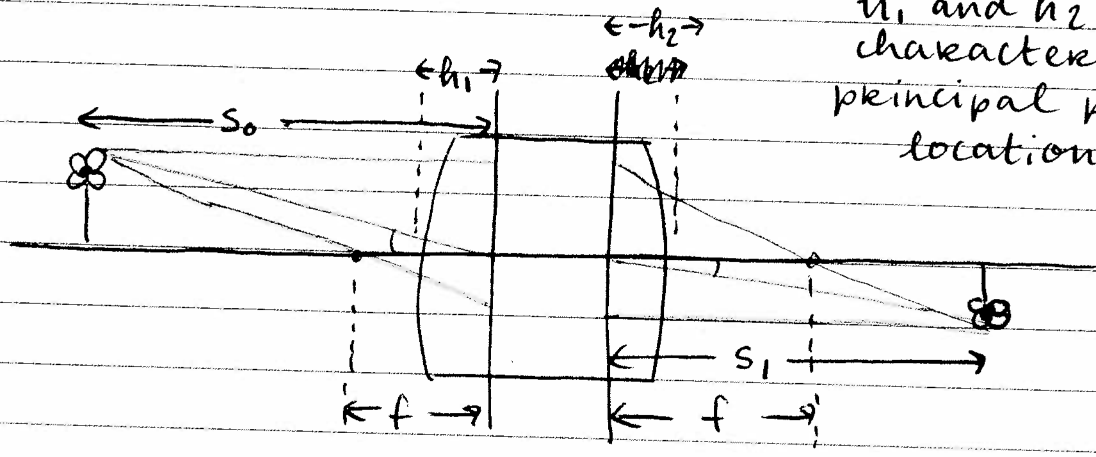
The principal point is there where a principal plane intersects the optical axis.

$$\frac{1}{S_0} + \frac{1}{S_i} = \frac{1}{f_{eff}}$$

measured from primary principal plane

measured from secondary principal plane

**Example**: For a thick lens with radii of curvature $R_1$ and $R_2$, index of refraction $n_\ell$, and thickness $d$, it is not hard to show:




$h_1$ and $h_2$ characterize the principal plane locations

effective focal length $\quad \frac{1}{f} = (n_\ell - 1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_\ell - 1)d}{n_\ell R_1 R_2}\right]$

$$h_2 = \frac{-f(n_\ell - 1)d}{R_1 n_\ell} \quad < 0 \text{ in this image}$$

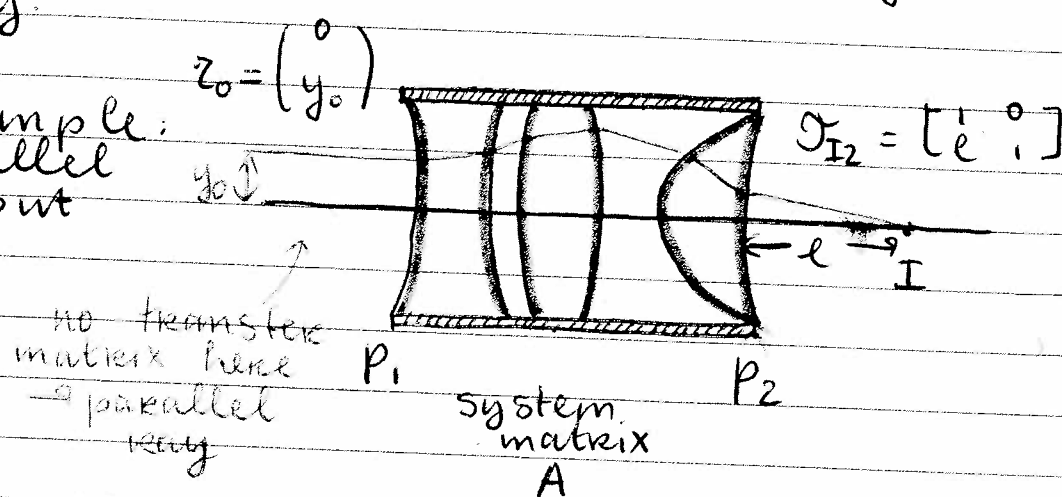
$$h_1 = \frac{-f(n_\ell - 1)d}{R_2 n_\ell} \quad > 0 \text{ in this image}$$

With these tools in hand you can now extract principal plane locations and $f_{eff}$ from a system matrix (how?).

## The system matrix

Even for complicated multi-lens systems it is straightforward to calculate the system matrix. To understand what the system matrix *means*, we can consider propagation of an incident ray.

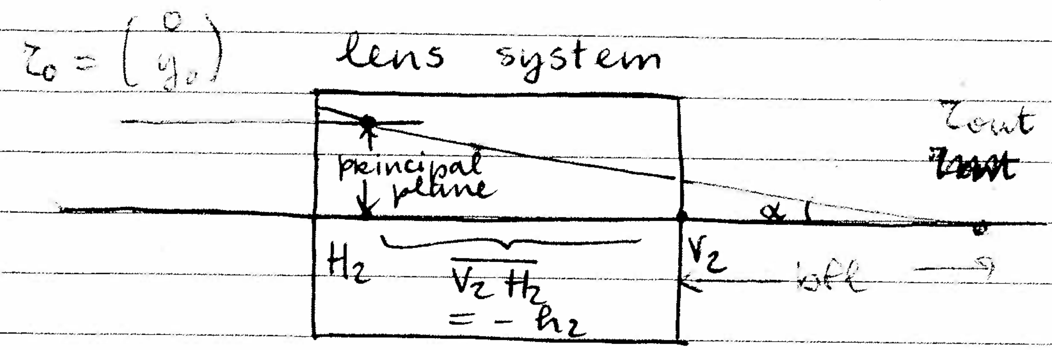
example: parallel input




The ray at point I is $r_{out} = \mathcal{J}_{I2} \cdot A \cdot r_0$

Can solve for the distance $\ell$ where $r_{out}$ is on the optical axis (i.e. $r_{out}[2] = 0$), which tells you the back focal length. Where $r_0$ and $r_{out}$ intersect tells you a principal plane.

A quick way to find the principal plane is to allow $\ell$ to take on both positive and negative values and solve for the value of $\ell$ for which $r_{out}$ is at height $y_0$




Note: you can also treat mirrors with transfer matrices:

$$M = \begin{bmatrix} -1 & -2n/R \\ 0 & 1 \end{bmatrix}$$ for a spherical mirror in a medium of index $n$

# Phys 434 – Lecture 3

## Aberrations

### 1.) Introduction

Up to now, we have only considered ray optics in the paraxial approximation. While the quantitative theory beyond this first-order theory is beyond the scope of this course (and it is typically dealt with numerically these days), we can gain a qualitative understanding of the ways in which real optical systems depart from the idealised behaviour of Gaussian optics.

Any departures from the idealised conditions are called aberrations. These are typically separated into chromatic aberrations (arising from the fact that $n$ is frequency/colour dependent) and monochromatic aberrations. The latter occur even for light of single frequency and are captured by including the next-order corrections in the expansion

$$\sin\varphi \approx \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots$$

The first order resulted in the paraxial approximation, whereas the 'third-order' theory captures five primary monochromatic aberrations (MA) also referred to as 'Seidel aberrations'. These will either make the image unclear or cause it to be distorted. We will focus on those five ΠA first and note that they can all occur in different combinations as they are of different physical origins.

## 2.) ΠA – Spherical aberration

In Lecture 5, we introduced spherical lenses and derived an expression that relates the two conjugate points S and P (a distance $s_o$ and $s_i$ from the interface, respectively) in the paraxial approximation:

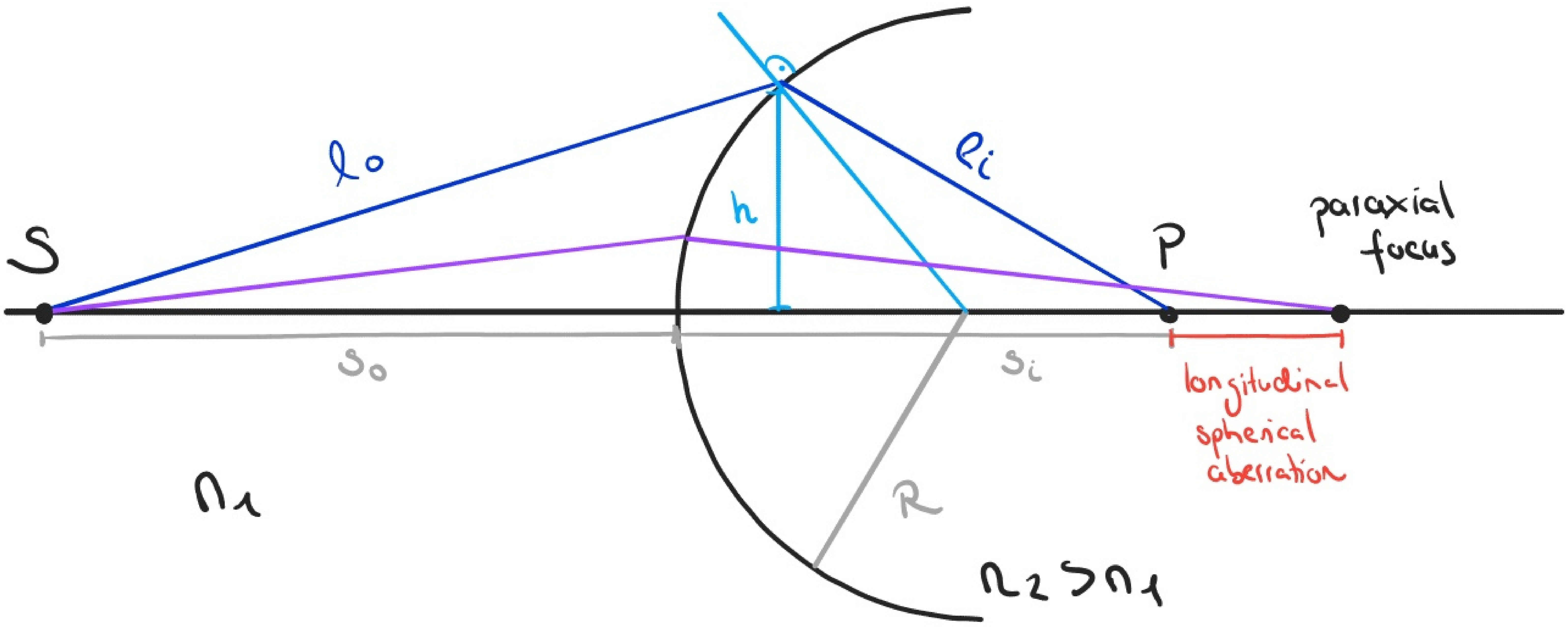
$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} .$$

R is the lens curvature radius

Accounting for the $\varphi^3$-term in the $\sin\varphi$ expansion, we can re-derive the above expression to obtain

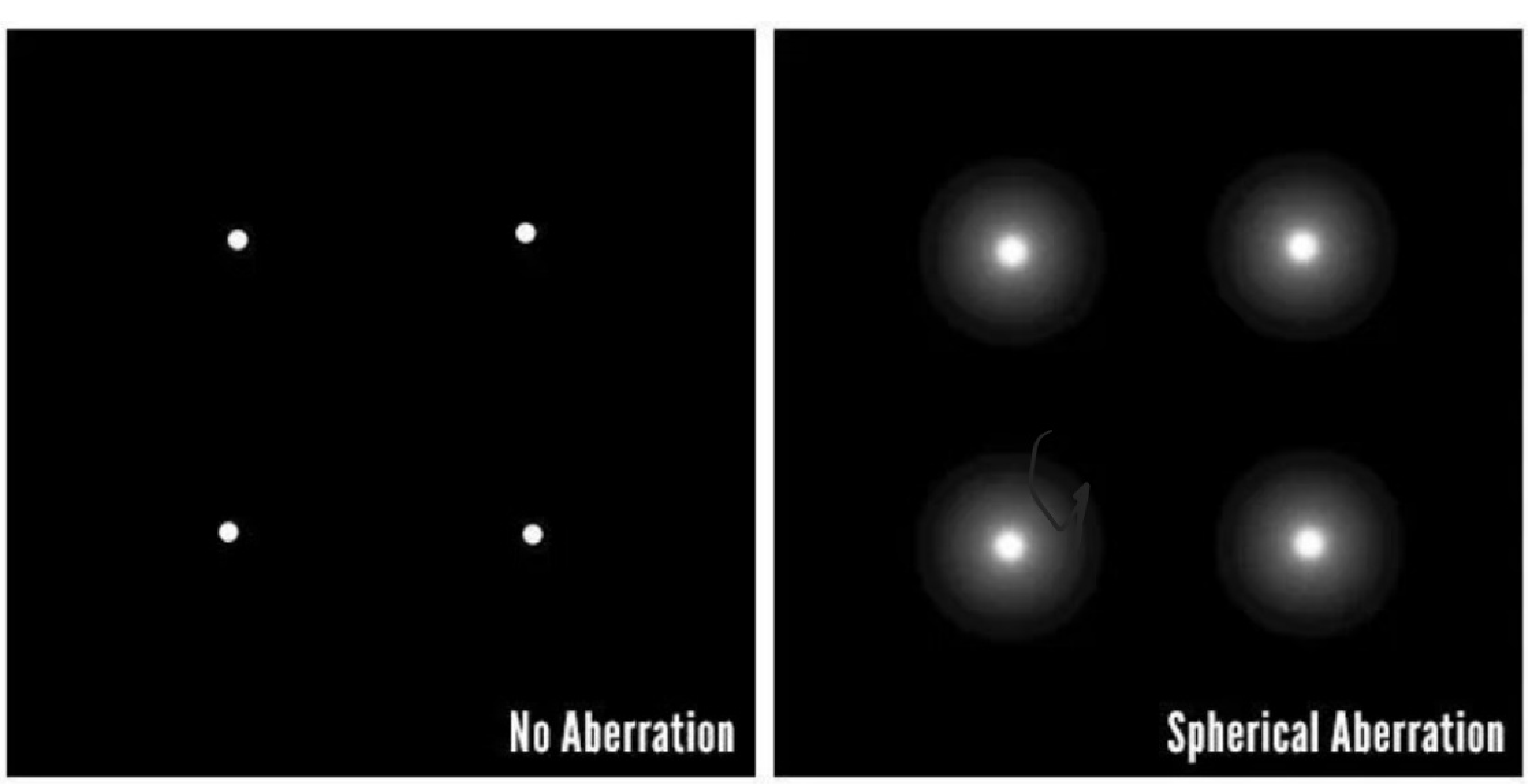
$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right],$$

where h is the distance from the optical axis (see geometry on next page) and the corrections to the first-order theory go as $h^2$. Thus, the position of the focus depends on where the ray hits the interface, and for larger h, rays will be focused closer to the point V. Spherical aberration

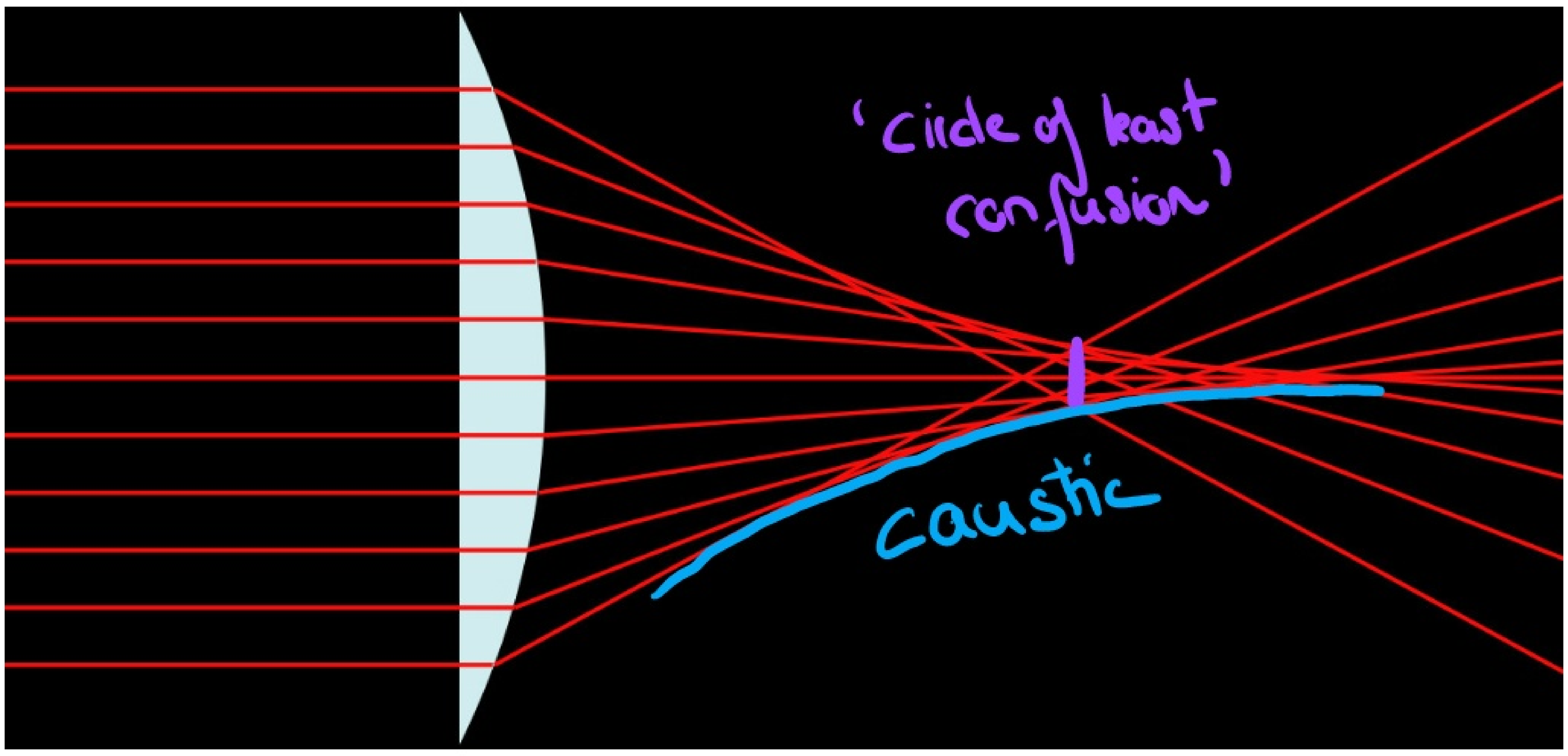



thus describes the fact that on-axis points are not being imaged onto a single image point. Hence, at the paraxial focus the peripheral rays will be out of focus, resulting in a 'halo' around a central spot (possibly with rings due to diffraction). The distance between the paraxial focus and the intersection of the peripheral rays is called the longitudinal spherical aberration, which is defined to be positive for a converging lens. For the diverging lens, the marginal rays typically intersect the axis right of the paraxial focus and the longitudinal spherical aberration is negative.

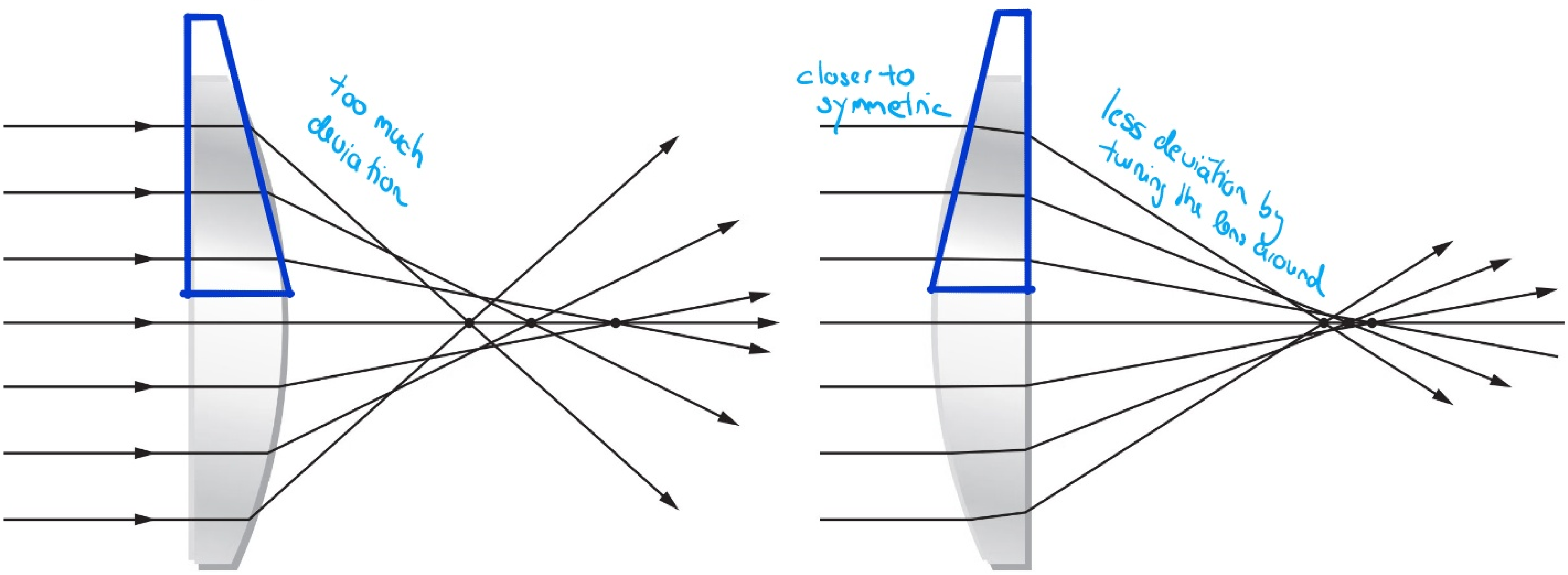
halos around 'point source'



Illustrating this behaviour for a number of rays, we observe that the refracted rays trace out a characteristic shape that is referred to as a 'caustic'. Note that the intersection of this caustic with the rays marks the 'circle of least confusion', i.e. the position where the image halo would have the smallest diameter. Here, the transverse spherical aberration (the height above the optical axis, where a given ray strikes the screen) would be minimal. This is typically the best place to observe the image.




The spherical aberration can be significantly reduced if the lens parameters $R_1$ and $R_2$ are choosen accordingly. This can be illustrated by looking at the rays that are far away from the optical axis; these rays are bent too much. As the top and bottom of the lens are essentially prisms, we can reduce the deviation of the rays by taking into account that the 'minimum' deflection angle corresponds to the symmetric scenario, where the angle of the incident ray and the emerging one are identical. We can thus significantly reduce the spherical aberration by simply turning the lens around (see below). In practice in the lab, one uses plano-convex lenses pointing the flat side towards the focal point and the curved side towards the parallel beam.

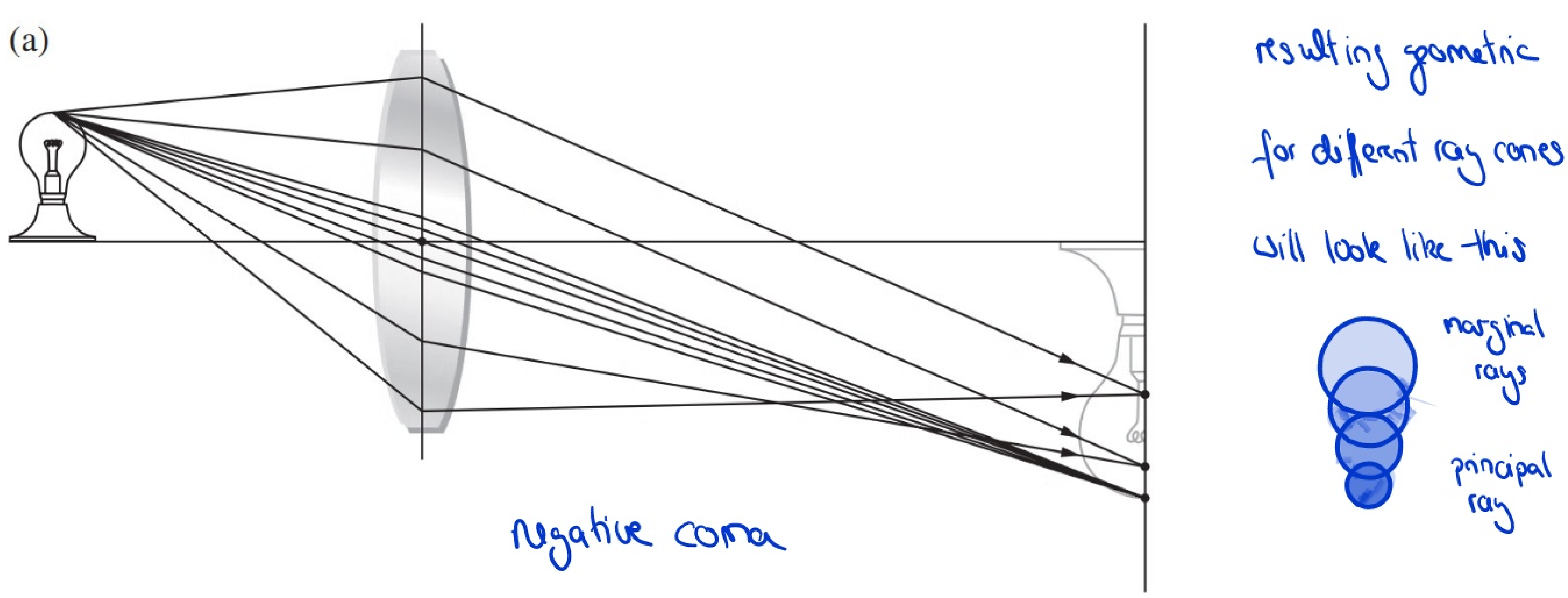



Additional strategies to reduce spherical aberrations :

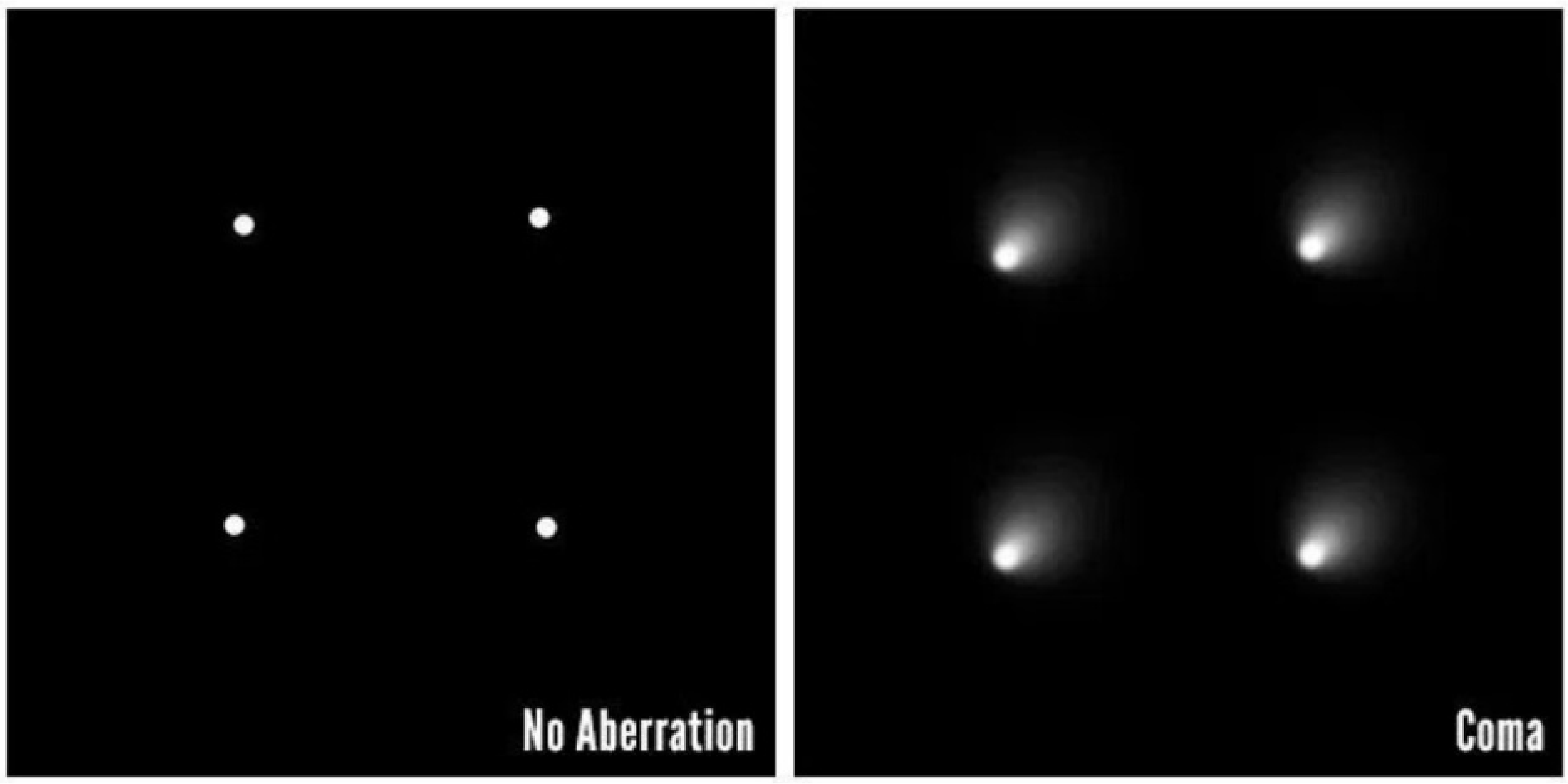
- lens shape : while for a parallel incident ray, plano-convex lenses are used, imaging a system with $S_o = S_i = 2f$ is achieved with bi-convex (two curved surfaces) lenses that have $R_1 = R_2$
- apertures : eliminating non-paraxial rays reduces the spherical aberration but also reduces the amount of light entering the system
- compensation plates : aspheric surfaces designed to reduce or compensate for spherical aberrations (equivalent to wave-front shaping); also possible to use gradient-index materials to construct lenses with very little spherical aberration

# 3.) ΠA - Coma

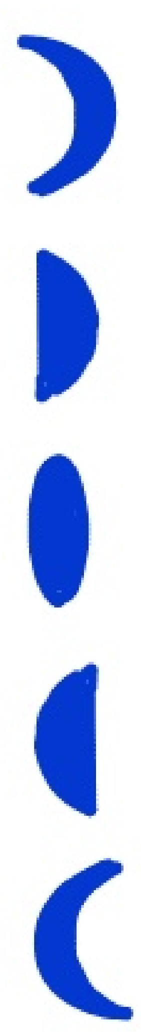
While spherical aberrations affect on-axis and off-axis points, the latter will also experience and additional aberration. It has its origin in the fact that the principal planes, introduced in Lecture 8, are only 'planes' in the paraxial approximation and generally principal curved surfaces. Because of this curvature, the source & the image distances, and hence the transverse magnification depends on the height of the ray at the lens. The fact that the effective focal lengths differ for rays traversing off-axis regions of the lens is illustrated below. For negative coma, the marginal rays have a smaller transverse magnification than the principal ray (the ray that passes through the principal points), while a positive coma corresponds to a larger transverse magnification for marginal rays.




Constructing a geometric image of the coma by following rays along different cones shows that these get imaged onto off-centred circles. This leads to a 'comet-like' tail that gives coma its name. This aberration can be further complicated by interference of the different 'circles'.




As for the spherical aberration, the lens shape also affects the coma:



strongly concave — for object at $\infty$, large negative coma

planar-convex

equiconvex

convex-planar

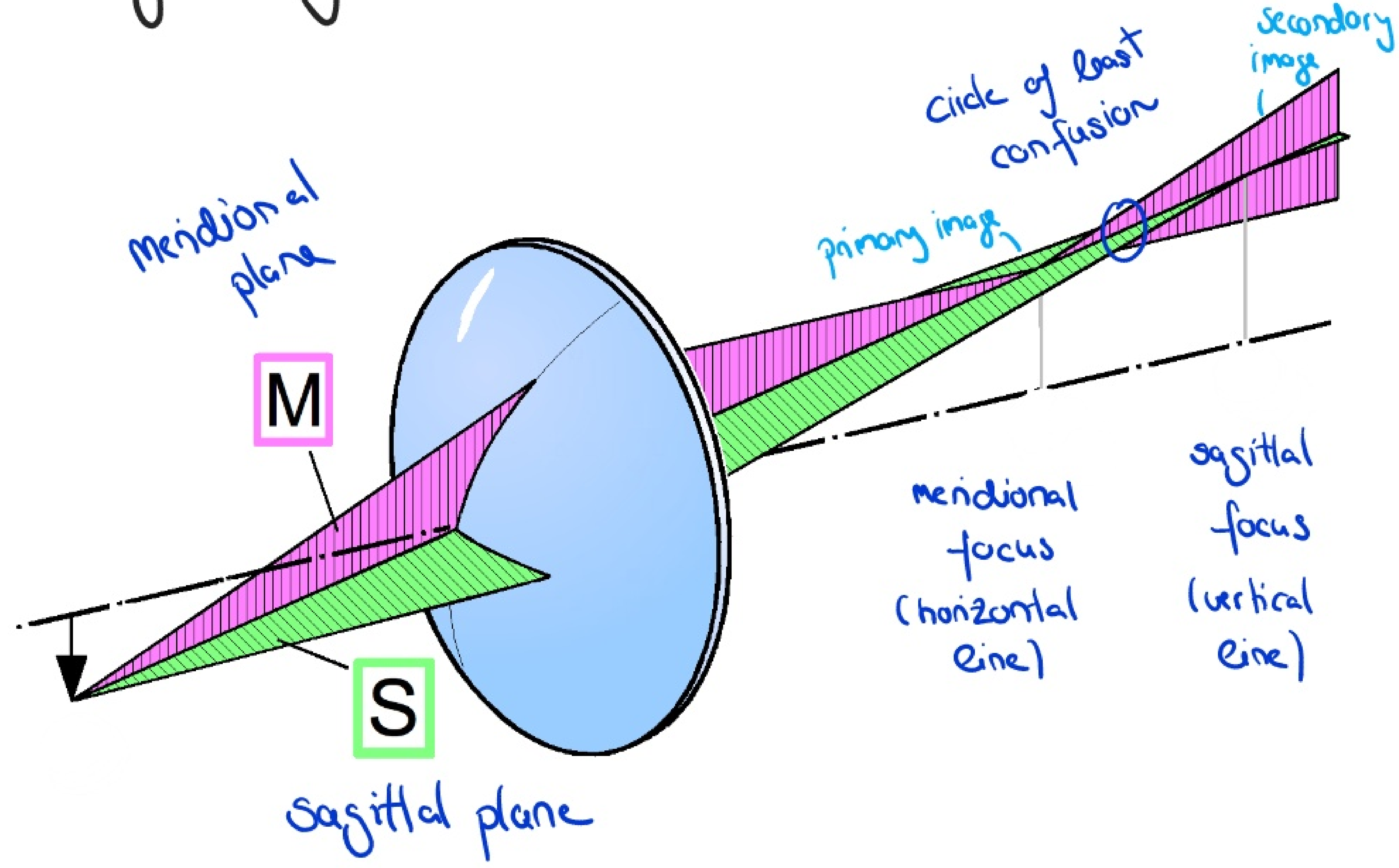
strongly convex — for object at $\infty$, large positive coma

$\downarrow$ coma increases and is zero somewhere

The fact that the coma can be made exactly zero for a given object and image distance using a single lens is important. For $s_o \to \infty$ this particular shape is almost planar-convex, which was also good to minimise the spherical aberration. Note however that a lens that is well corrected for a specific set of $s_o, s_i$ will not perform well for a different set $s_o, s_i$.

4.) MA - Astigmatism

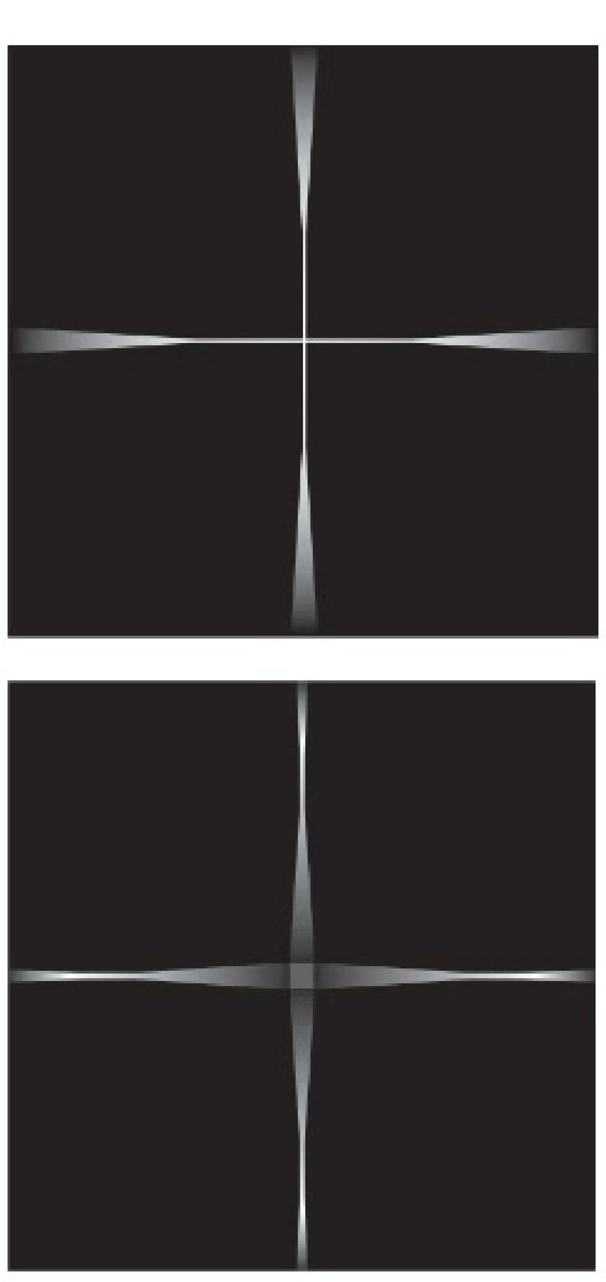
Note that this is different from the visual astigmatism and occurs even for perfectly spherical lenses. This aberration occurs when an object point is positioned a considerable distance from the optical axis and the incident cone of rays will strike the lens asymmetrically. To describe this aberration one typically defines two planes, the meridional (or tangential plane, which contains the primary ray and the optical axis, and the sagittal plane, which contains the primary ray and is perpendicular to the meridional plane. This geometry is illustrated below.




The rays in these two planes come to focus at different points as they encounter the lens at different angles. This introduces a biaxial symmetry to the image of a point. Unlike the previous two aberrations, astigmatism does not depend on the shape of the lens, just its dioptric power and the angle of the incident rays. This makes it difficult to reduce this aberration.

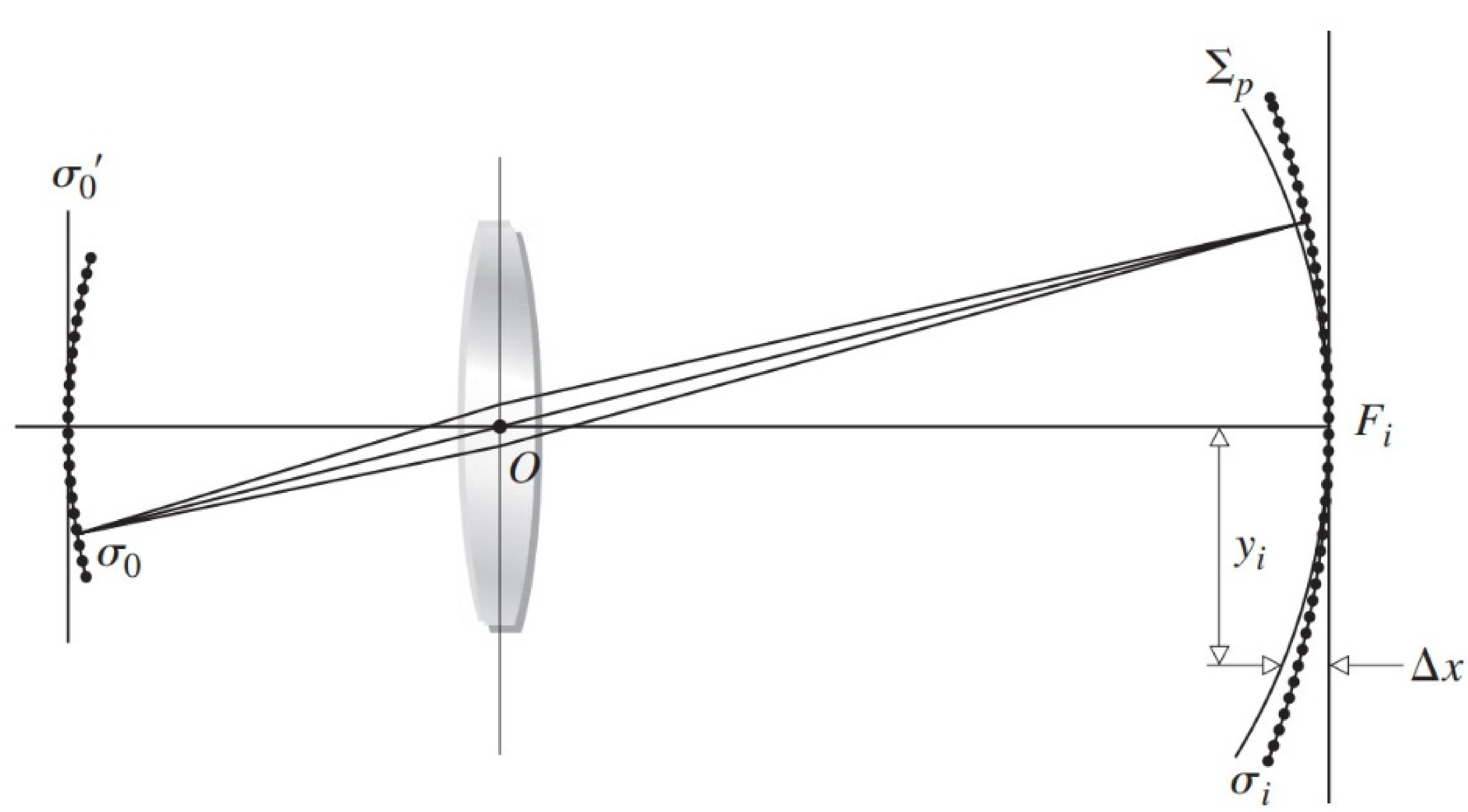

## 5.) MA – Field curvature

Suppose we had an optical system that was free of all the aberrations discussed so far. This would imply that each object point would correspond to a single point on the image surface. We assumed before that this is a plane, an assumption that only holds in the paraxial approximation. At finite lens apertures, however, the resulting image surfaces are curved and the focal points are different for marginal rays and those close to the optical axis as illustrated below; in the paraxial image plane the centre will be in focus, while moving the screen closer will bring the edges into focus.



paraxial image plane

moving screen closer



Consider a planar object being imaged onto a curved surface. The resulting shape will be parabolic and is also known as the 'Petzval surface'. Note that it will curve toward the object plane for a convex lens and away from it for a concave lens. Combining different positive and negative lenses one can eliminate the field curvature. For a system of $N$ thin lenses, the displacement $\Delta x$ of an image point at $y_i$ (see last page) reads

$$\Delta x = \frac{y_i^2}{2} \sum_{j=1}^{N} (n_j f_j)^{-1} ,$$

where $n_j$ and $f_j$ are the refractive index and focal length of the $j$th lens. $\Delta x$ will thus be constant even if the shapes or positions of individual lenses are altered provided $n_j$ and $f_j$ are fixed. For two lenses, we can obtain $\Delta x = 0$, when

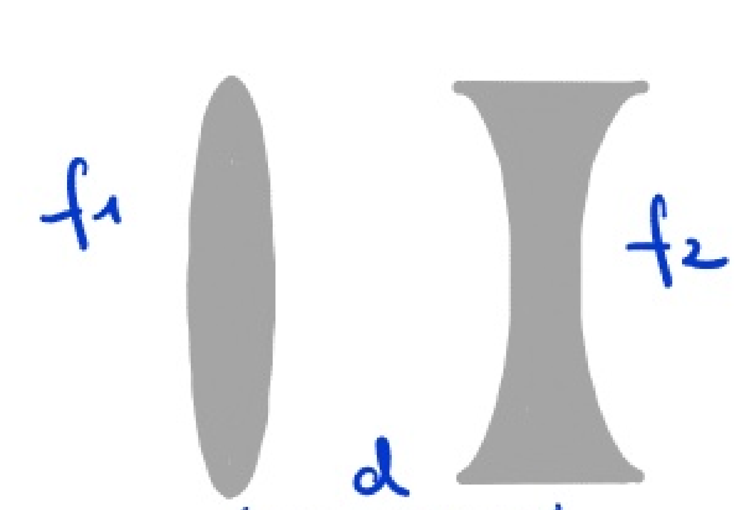
$$\underline{\underline{n_1 f_1 + n_2 f_2 = 0}} .$$

Petzval condition

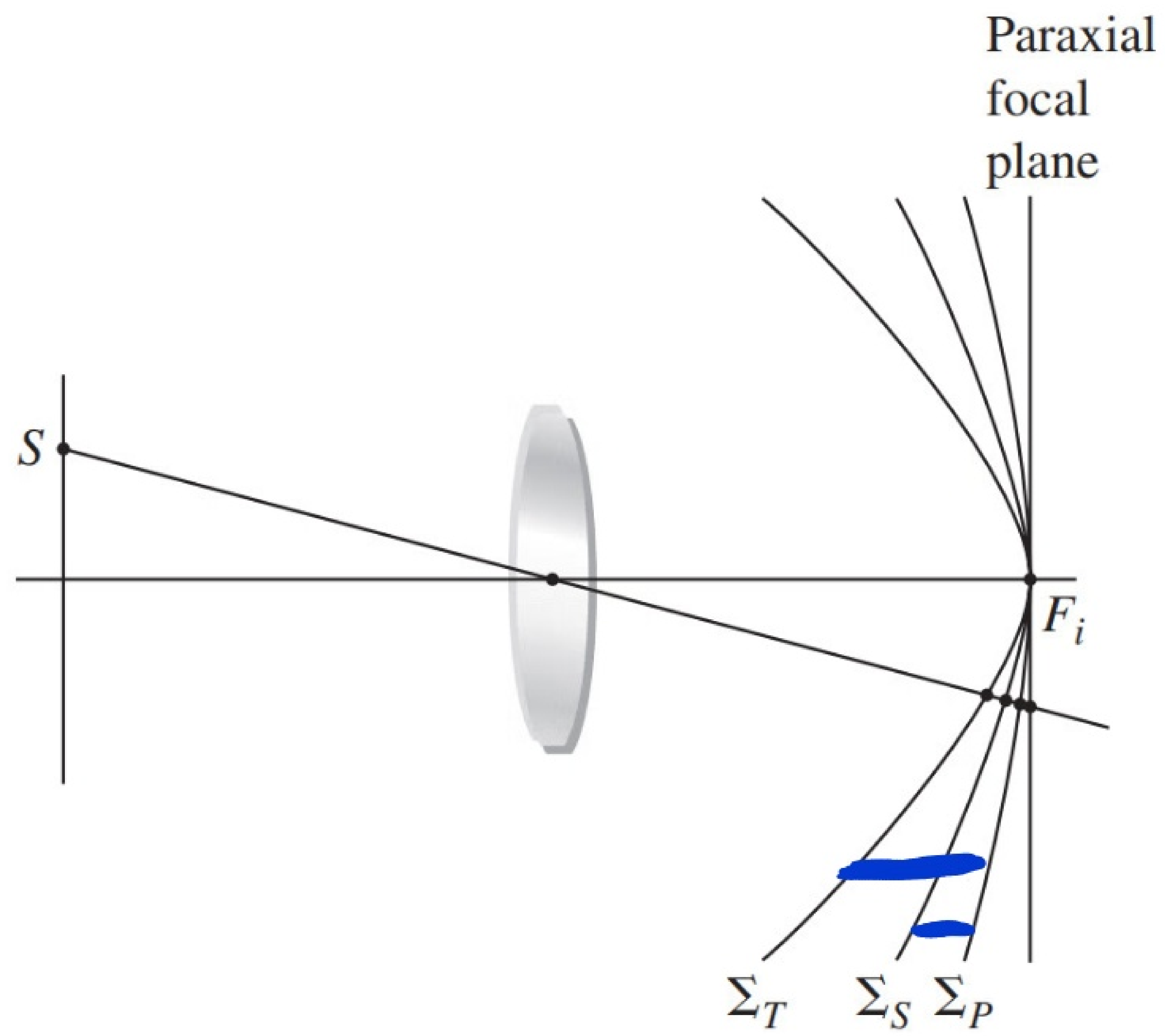
For two lenses, separated by a distance $d$ and $f_2 = -f_1$, $n_1 = n_2$, we can calculate for the effective focal length $f$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{d}{f_1^2} \Longrightarrow \underline{\underline{f = \frac{f_1^2}{d}}} .$$

The system thus has an overall positive focal length and a flat field.

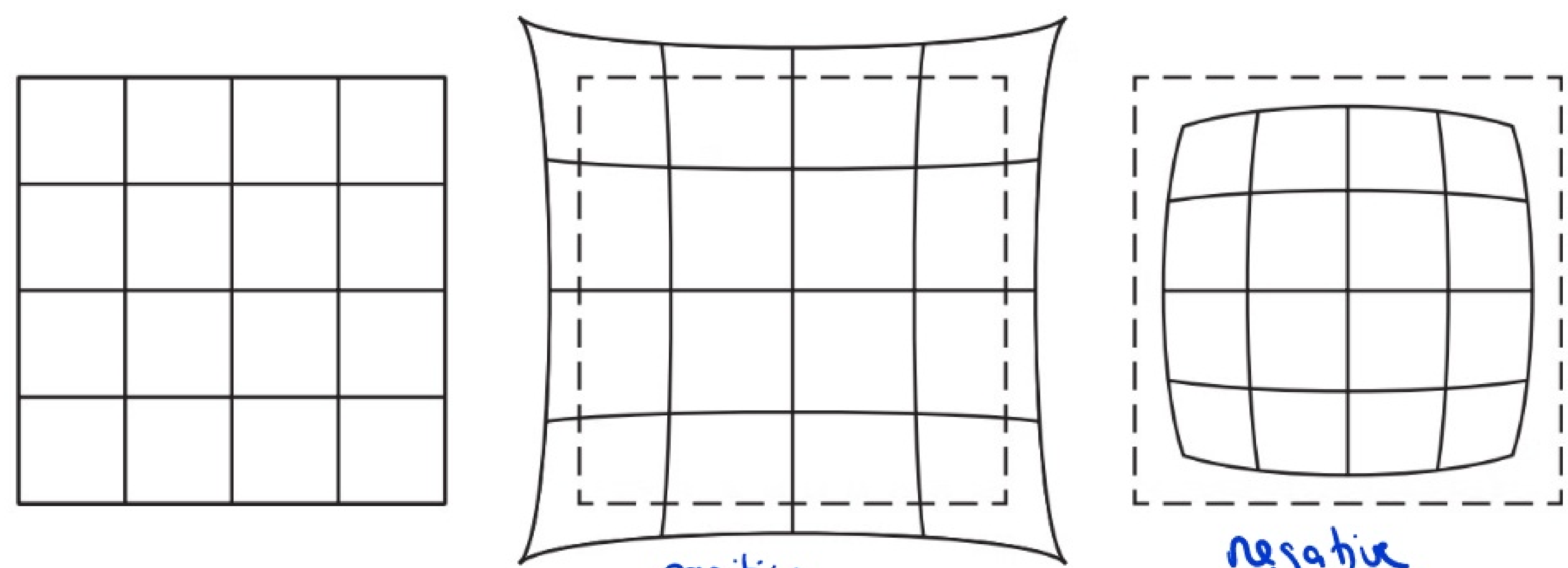


Astigmatism and field curvature closely interact with each other. When astigmatism is present, then there are two paraboidal image surfaces, one for rays in the sagittal plane and one for those in the meridional plane. Those will be both on the same side of the Petzval surface but $\Sigma_{i\,\text{tangential}}$ is always three times further away from $\Sigma_{i\,\text{petzval}}$ than $\Sigma_{i\,\text{sagittal}}$. In the absence of astigmatism, $\Sigma_{iT}$ and $\Sigma_{iS}$ coalesce with $\Sigma_{iP}$.

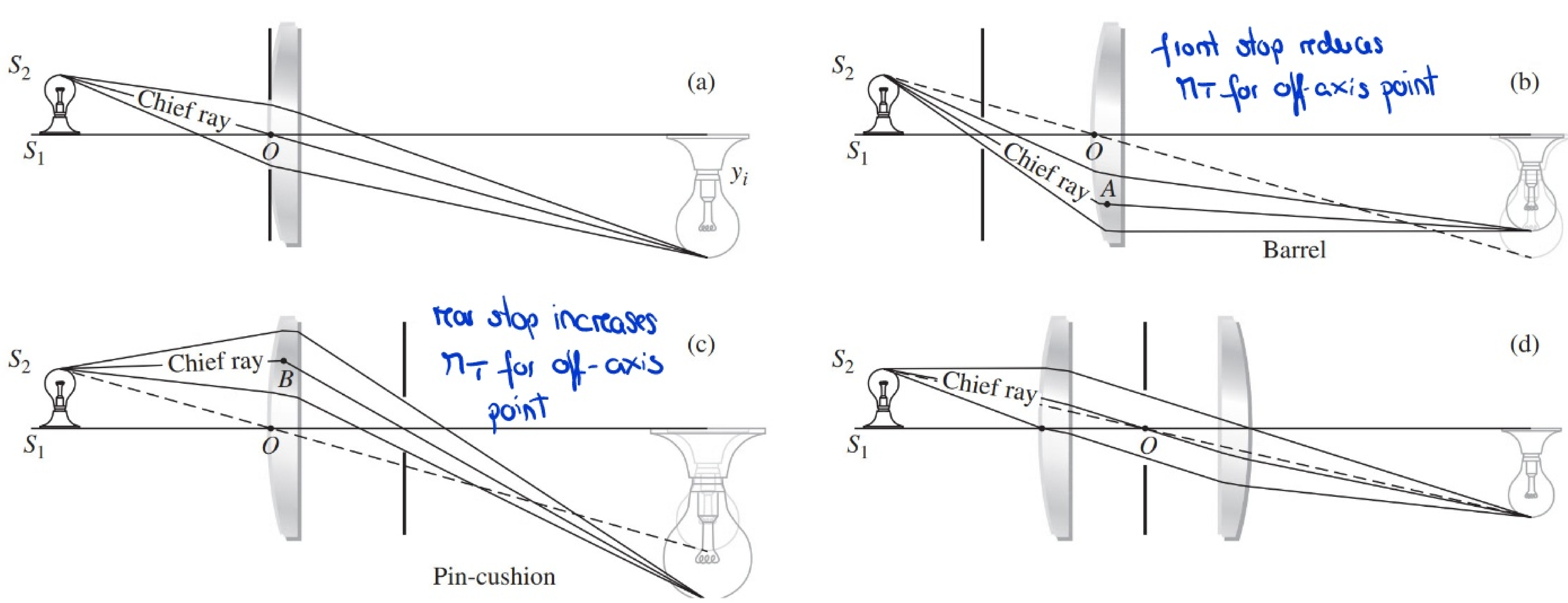



## 6.) MA - Distortions

This aberration arises because the transverse magnification can depend on the image distance from the optical axis, leading to a distortion of the image. In other words, this aberration arises when rays sampling different regions of the lens experience different focal lengths. In the absence of other aberrations, each point is individually in focus but the entire image misshapen. Consequently, a grid can be deformed in the following ways




Distortion can be eliminated by adding an aperture stop midway between identical elements as the distortion from the first lens will exactly cancel the contribution from the second lens. In the perfect symmetric case, we have $M_T = 1$, but even for $M_T \neq 1$ a midway stop reduces distortion.

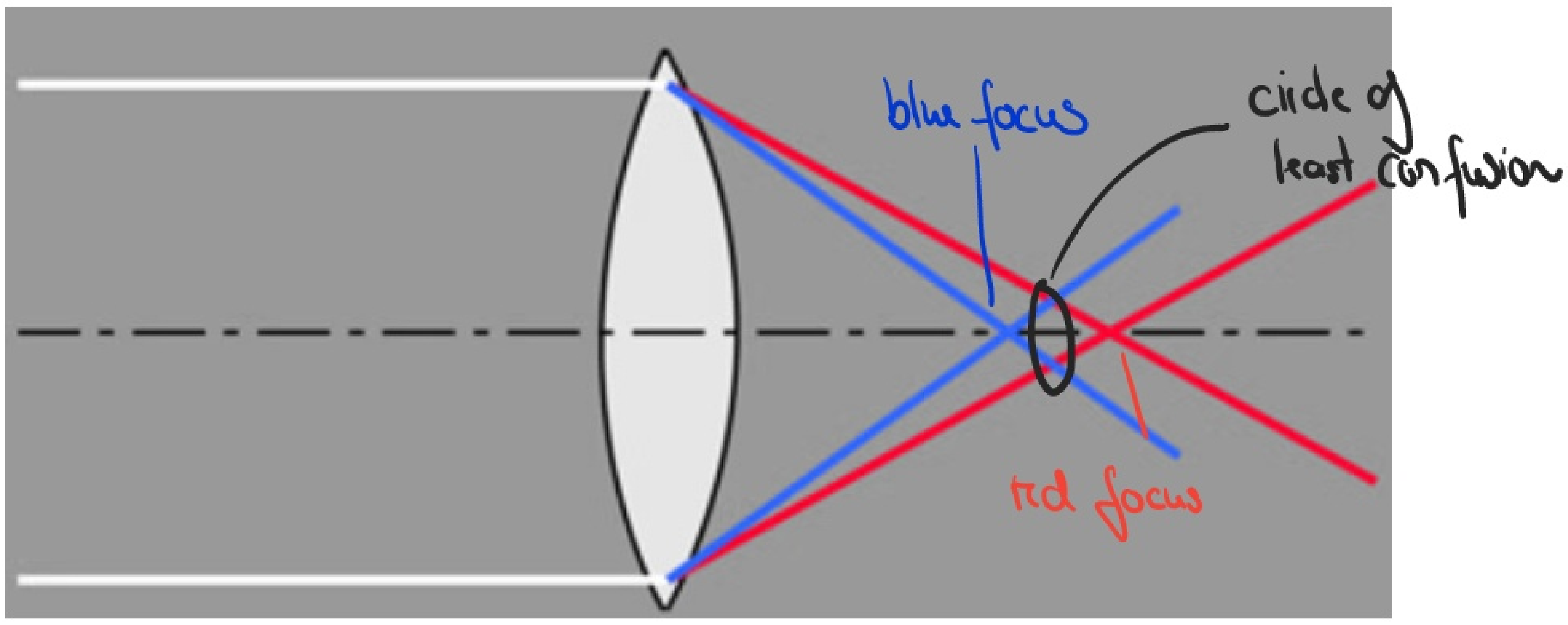


## 7.) Chromatic aberrations

The five Seidel aberrations have been discussed in terms of monochromatic light. For non-monochromatic light, however, the fact that $n$ varies with wave length can completely outweigh the five Seidel aberrations. The fact that 'different coloured' rays will traverse different paths results in chromatic aberrations. Consider the thin lens equation

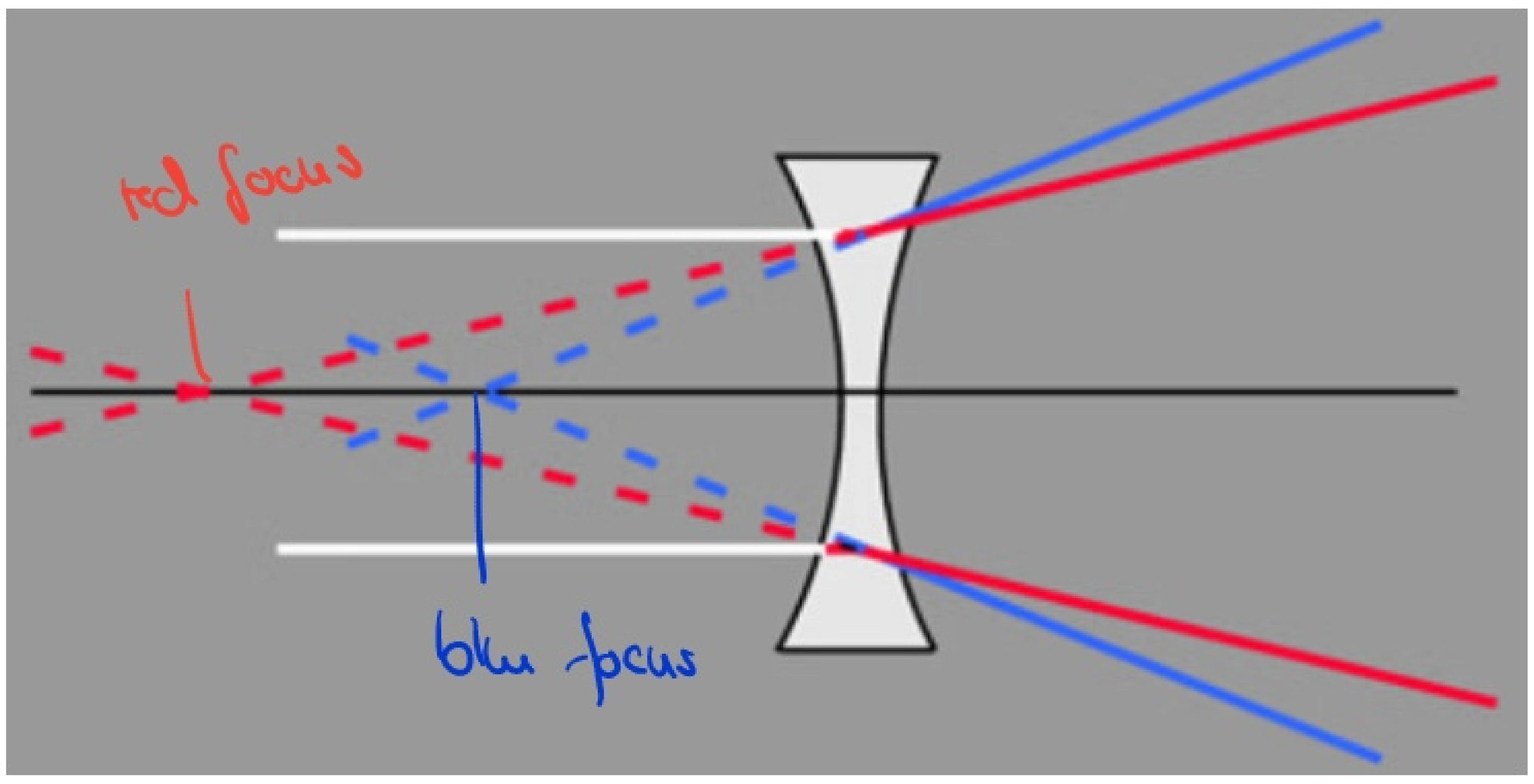
$$\frac{1}{f} = (n(\lambda)-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$

which dictates that the focal length changes with frequency / wavelength, i.e. $f$ increases with $\lambda$, as $n(\lambda)$ decreases with $\lambda$. This is illustrated here




(drawn hugely exaggerated)

We can combine these two effects to largely cancel chromatic aberrations by combining two lenses.



The combined system is called an 'achromatic doublet' and it is possible to design the lens system in such a way that the focal lengths for different colours are identical: