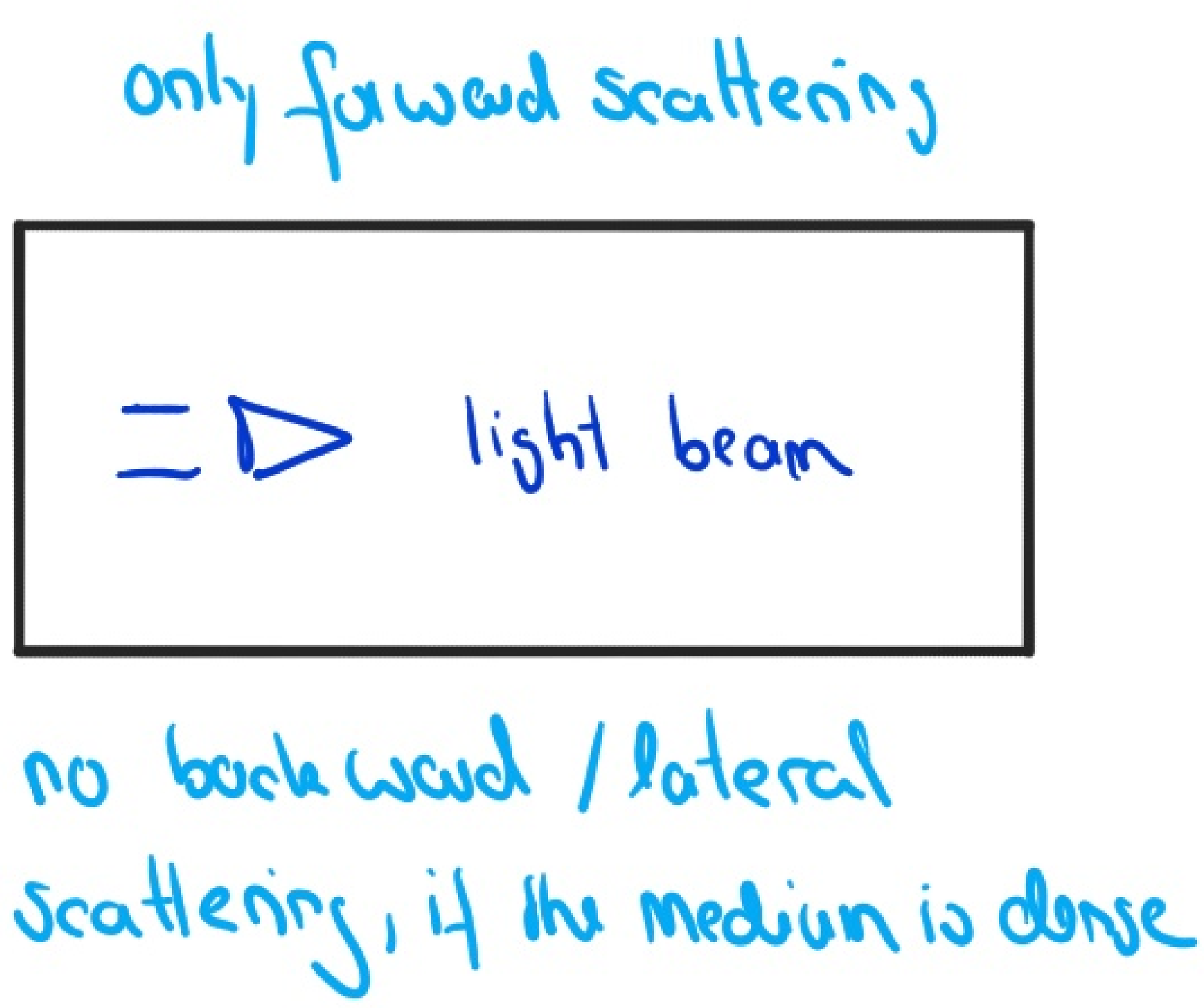
PHYS 434 - Lecture 3

Reflection, Refraction, Huygens & Fermat

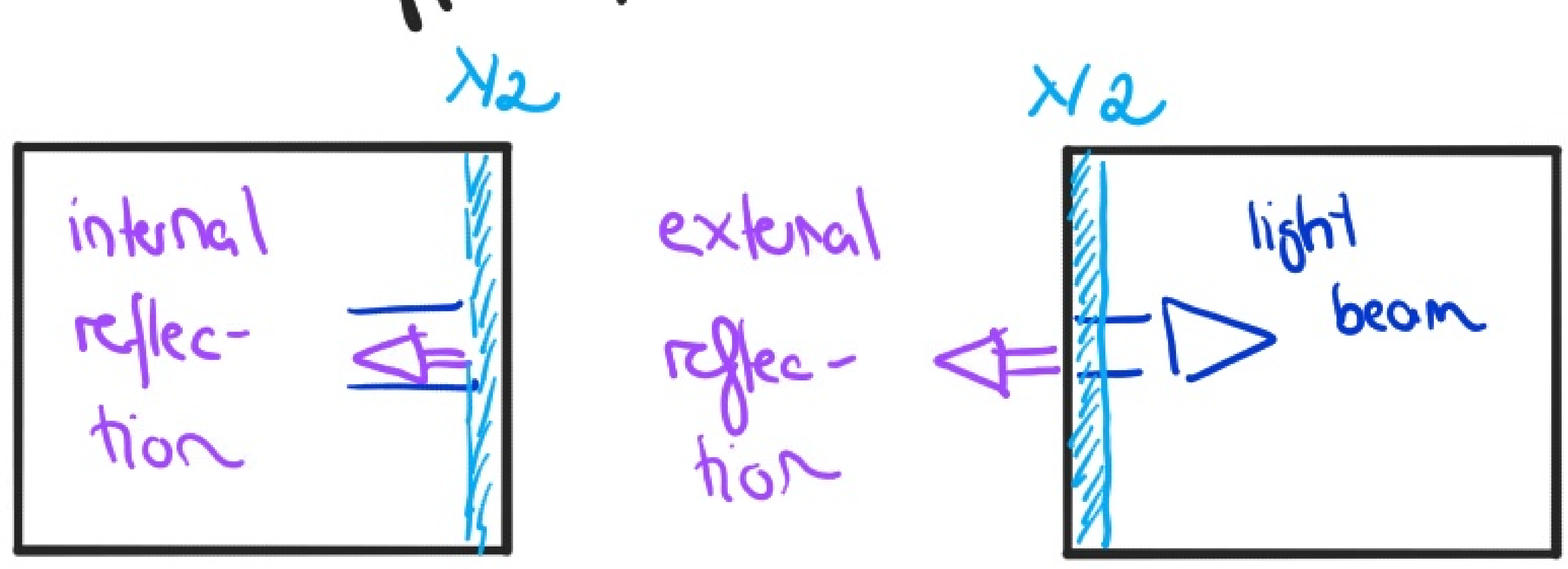
1.) Microscopic picture

To understand why refraction occurs at interfaces (i.e. materials of different refractive indices), we consider the picture introduced in Lecture 2, i.e. a medium consists of microscopic oscillators, that are stimulated by the incident wave to emit spherical waves. Take a large block of e.g. glass through which light is propagating.

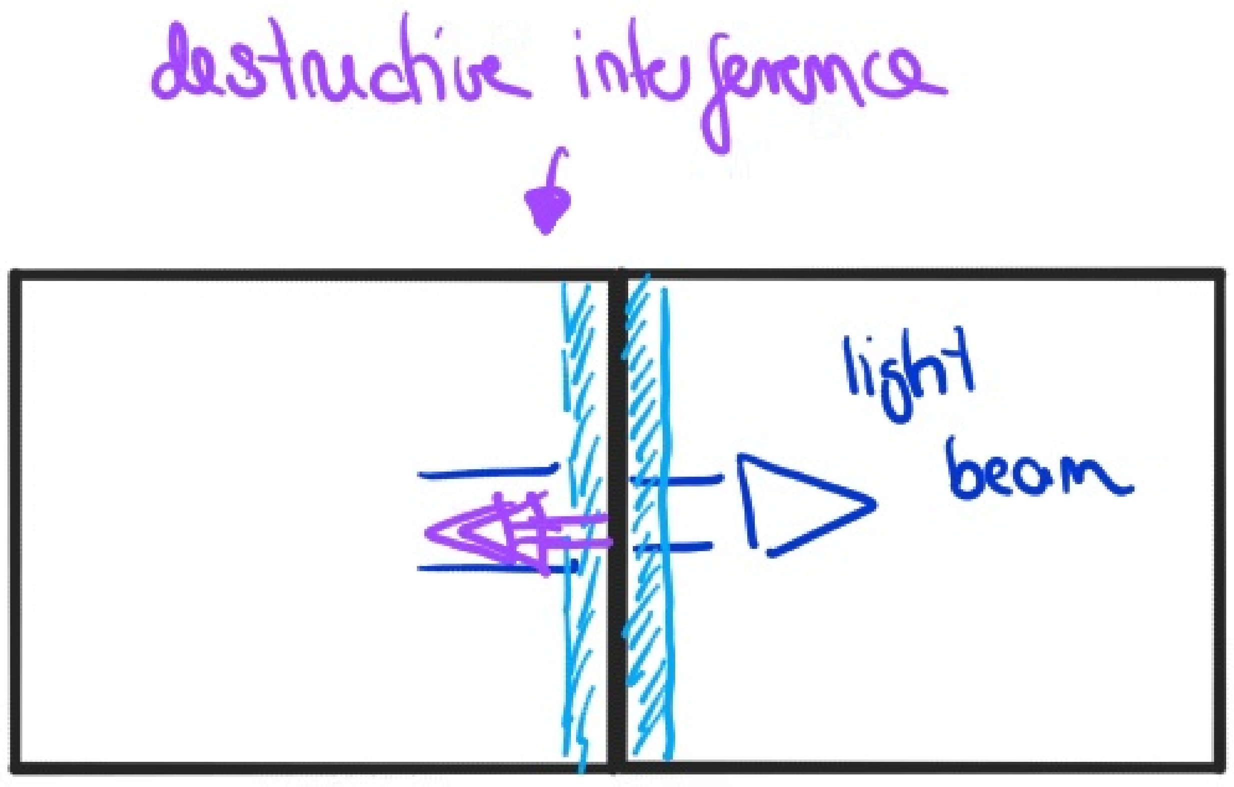


As discussed in the last lecture, the wavelets scattered off oscillators within λ of each other interfere with each other to prevent lateral / backward scattering.

Now consider what happens, when we cut the block in half:

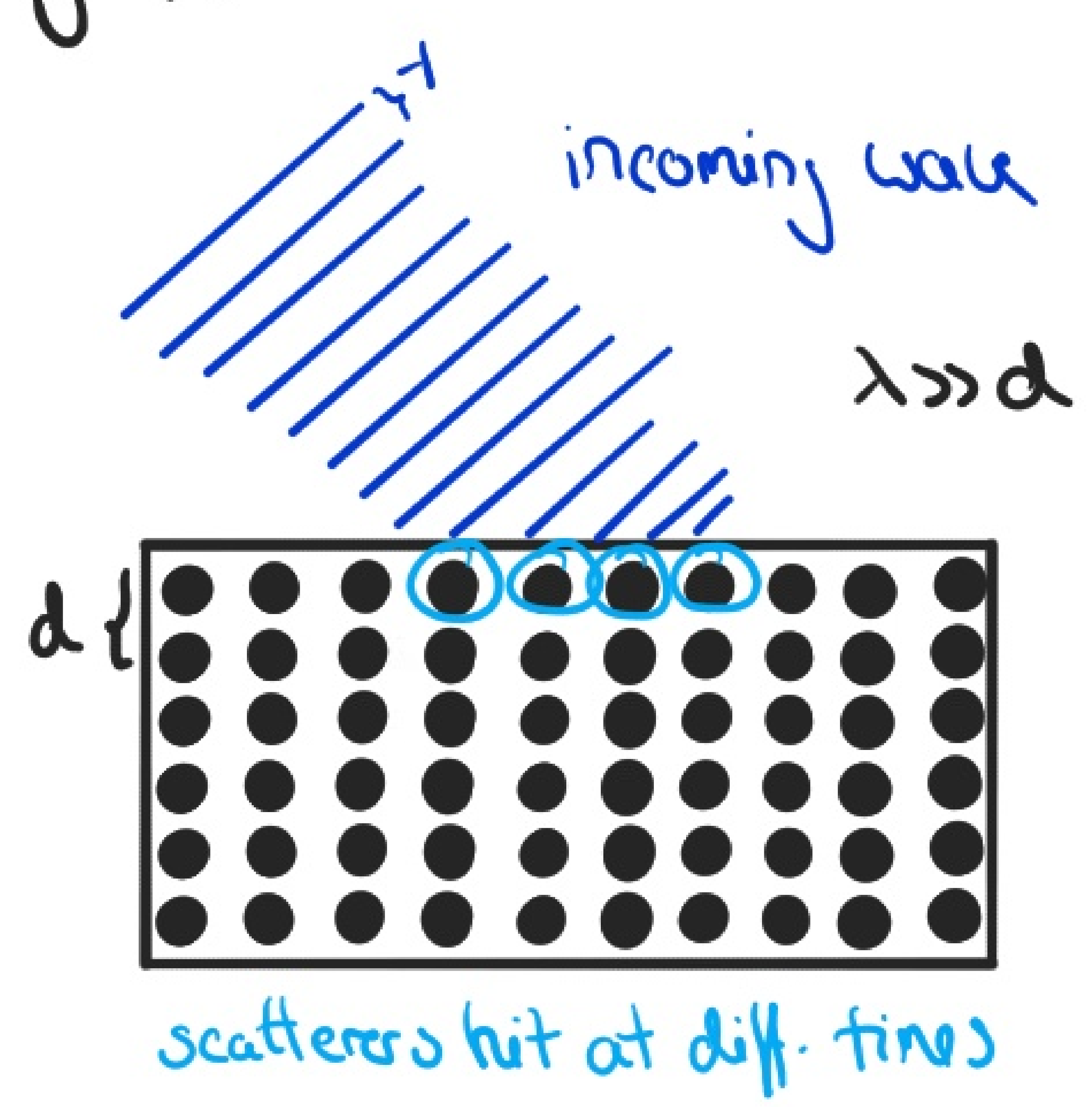


The oscillators within $\lambda/2$ of the interfaces become unpaired in the backward and forward direction and the backward propagating waves no longer cancel $\Rightarrow$ we expect fields to move backwards at each interface, which is called internal and external reflection, respectively. For an air-glass interface about 4% of the incident light is reflected, independent of the block's thickness (only the oscillators within a thickness $\lambda/2$ contribute to the reflection).

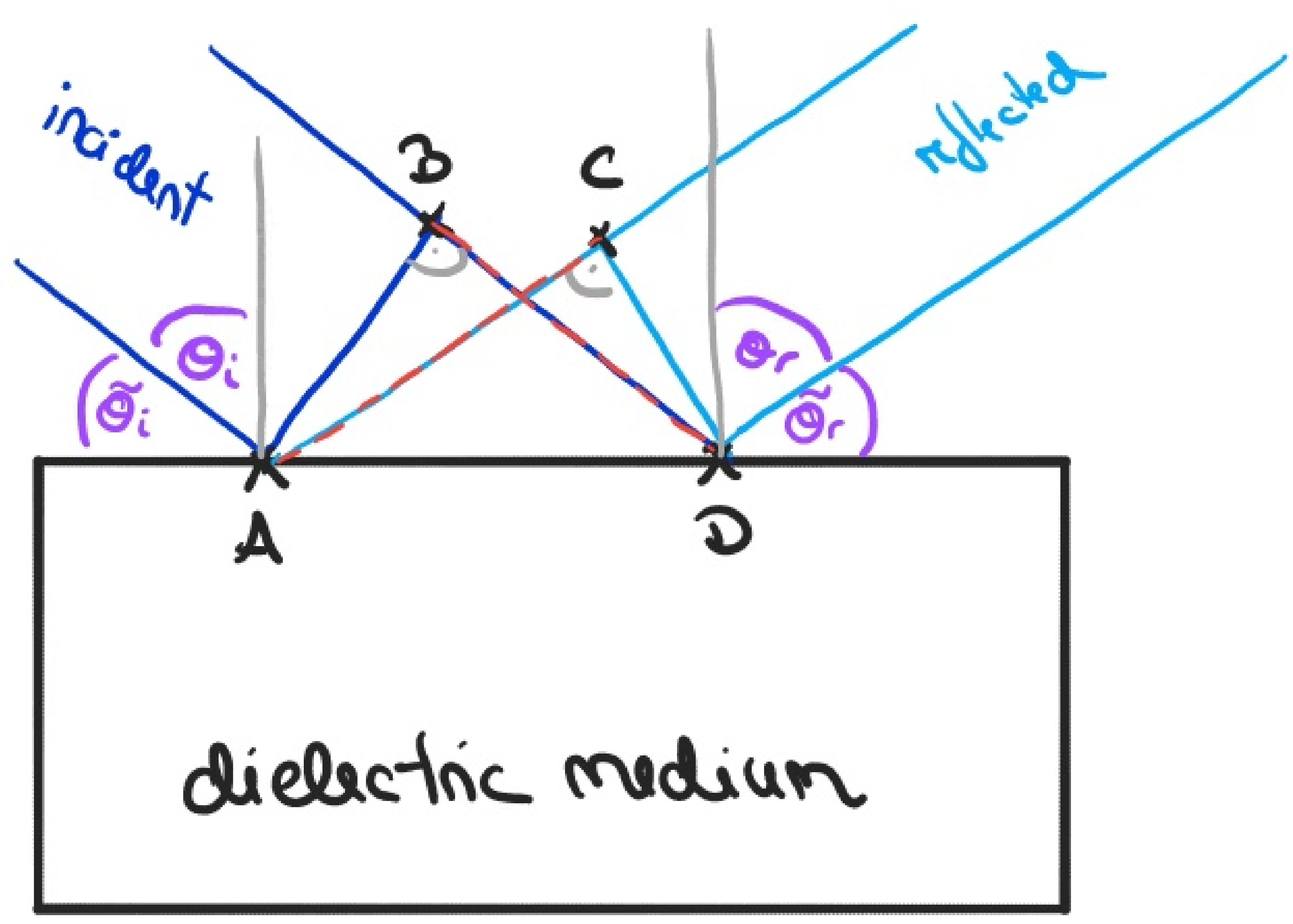


We can learn one more thing from this thought experiment: If we bring the two blocks back together, we expect the backward scattering to disappear $\Rightarrow$ the two reflected beams have to interfere destructively, which is only possible if there is a 180° relative phase shift between internally & externally reflected light.

2.) Law of reflection



Consider the oscillator / scattering picture to understand how a wave hitting an interface at an angle is reflected: Different oscillators see different phases of the incident EM wave, implying that they also oscillate with different phases. All of the waves emitted back into the direction of the incident wave will interfere but only in one specific direction will this interference be constructive (all the other directions will cancel out). The angle of the reflected wave will solely depend on the incidence angle and from everyday experience we know that both angles are equivalent. We can prove this by looking at the propagation of a specific wave front:



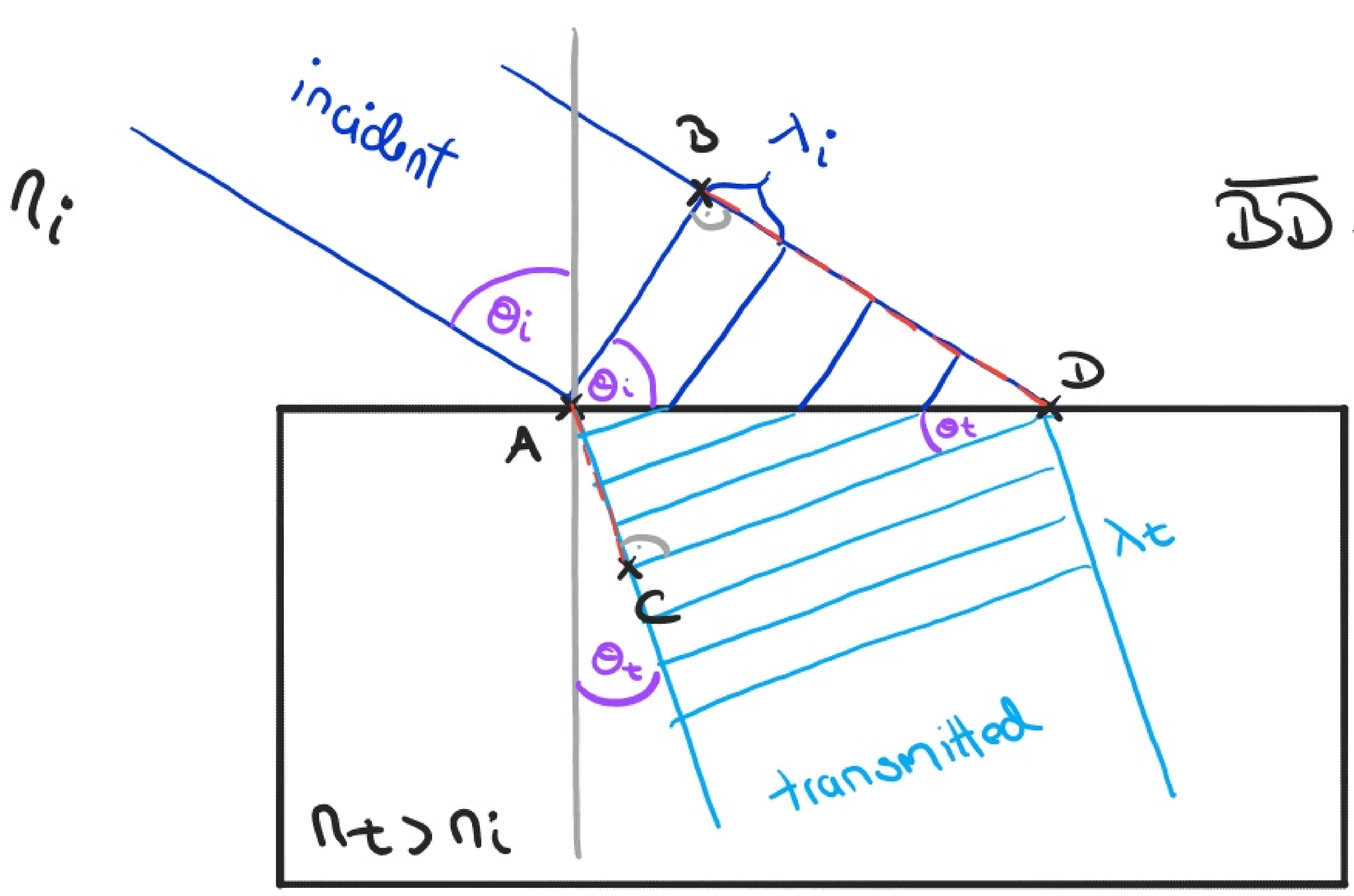
Consider points A and B that lie along the same wave front of the incident wave, which implies (by definition) that they have the same phase. Similarly C and D lie on the same wave front of the outgoing wave.

The oscillator at A is hit first and generates a wave that moves outwards. Upon reaching point C, this wave will have the same phase as the new wave, emitted by the oscillator at point D (because A and B are in phase). These waves thus interfere constructively and form a new wave front. As all waves outside the dielectric travel at the same velocity, the distance $\overline{AC}$ is equal to $\overline{BD}$. As the two right-triangles share the same hypotenuse $\overline{AD}$, the remaining two angles must be the same so that $\tilde{\theta}_i = \tilde{\theta}_r$ and thus also $\theta_i = \theta_r$.

Two additional notes : (i) Exact cancellation in all but one direction requires not only that a sufficient number of scatterers are present but also that the surface of the dielectric is sufficiently flat. Inhomogenieties would lead to additional directions, where interference causes non-zero intensity. The result is what we call diffuse reflection. (ii) By looking at the reflection process from an orthogonal direction, we also observe that the incoming and reflected beams lie in the same plane, the plane of incidence.

3.) Law of refraction

When light hits a medium, some fraction will be transmitted into the material in a direction that is determined by the angle of the incident wave and the response of the scatterers in the medium. We again follow a wave front $\overline{AB}$ as it propagates into a medium over the time interval Δt. So



$$\overline{BD} = v_i \Delta t \quad \&$$

$$\overline{AC} = v_t \Delta t$$

Looking at the two right triangles again, we find

$$\sin\Theta_i = \frac{\overline{BD}}{\overline{AD}} \quad \text{and} \quad \sin\Theta_t = \frac{\overline{AE}}{\overline{AD}}$$

eliminate $\overline{AD}$

$$\Rightarrow \quad \frac{\overline{BD}}{\sin\Theta_i} = \frac{v_i \cancel{\Delta t}}{\sin\Theta_i} = \frac{\overline{AE}}{\sin\Theta_t} = \frac{v_t \cancel{\Delta t}}{\sin\Theta_t}.$$

Finally substituting $v_i = c/n_i$ and $v_t = c/n_t$, we have

$$\underline{\underline{n_i \sin\Theta_i = n_t \sin\Theta_t}}.$$

Snell's law
or law of refraction

We can learn two more things from observing the wave front propagate into the medium:

(i) As the oscillators in the medium respond to the driving incident wave at the same frequency (at least for the linear media considered so far), the wave frequency in the medium is the same as outside. The change in n and v however implies

$$\lambda_t = \frac{v_t}{\nu} = \frac{v_t}{v_i}\cdot\lambda_i = \frac{n_i}{n_t}\lambda_i < \lambda_i.$$

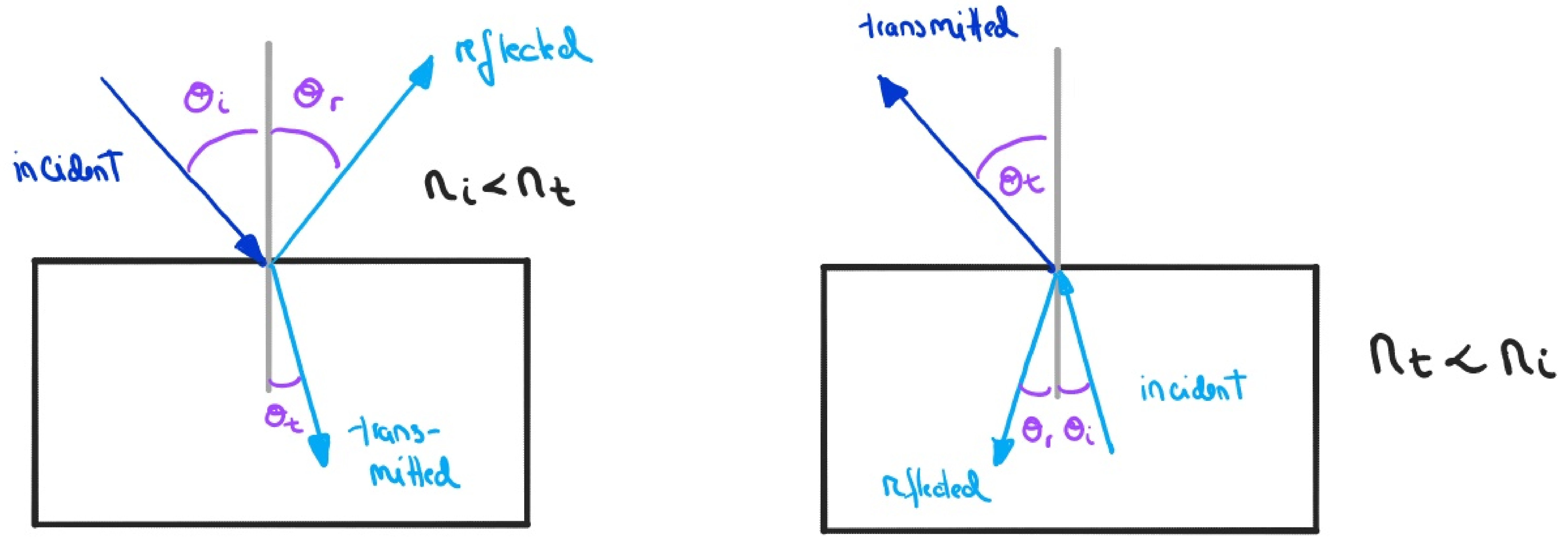
for $n_t > n_i$

If $n_t > n_i$, this implies that the wavelength decreases in the medium, which you can indeed observe in the sketch above.

(ii) The sketch also shows that $\overline{AB} < \overline{ED}$, implying that the beam is broadened in the medium if $n_t > n_i$. A broader cross-section implies a smaller energy density, i.e. lower intensity.

While the wave-front picture gives some insight into the underlying physics of reflection and refraction, both phenomena are typically formulated in terms of rays (lines pointing in the direction of the energy flow, which is perpendicular to all wave-fronts). Angles are subsequently measured relative to the surface normal. All the rays lie in the same plane, the plane of incidence.

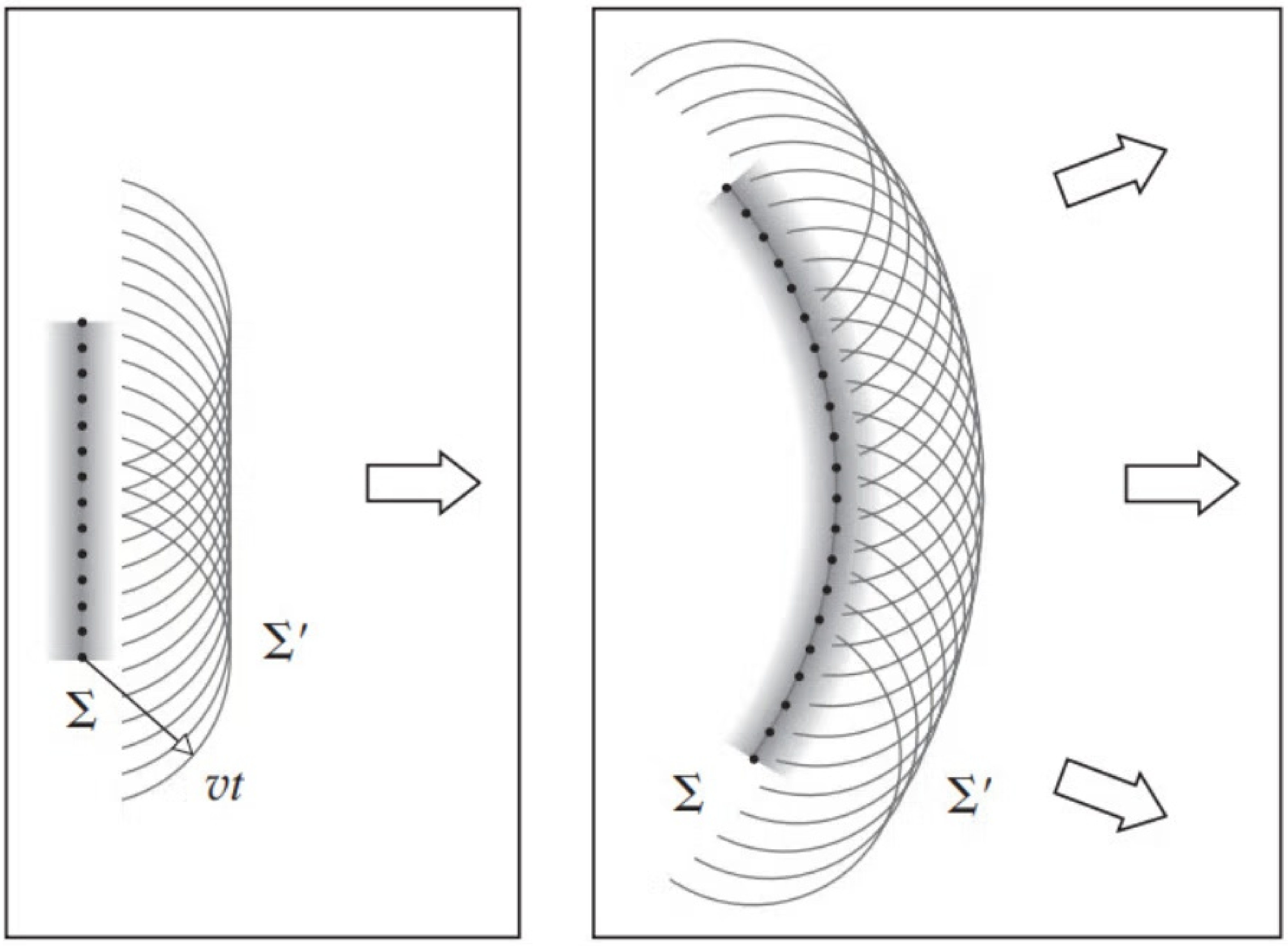
So far, we have discussed the scenario $n_t > n_i$ (e.g. light propagates from air into glass). In the inverse case $n_i > n_t$, the transmitted ray is no longer bent towards the normal, but away from it. Note that light will travel the same path, you can simply adjust the direction:



4.) Huygens' principle

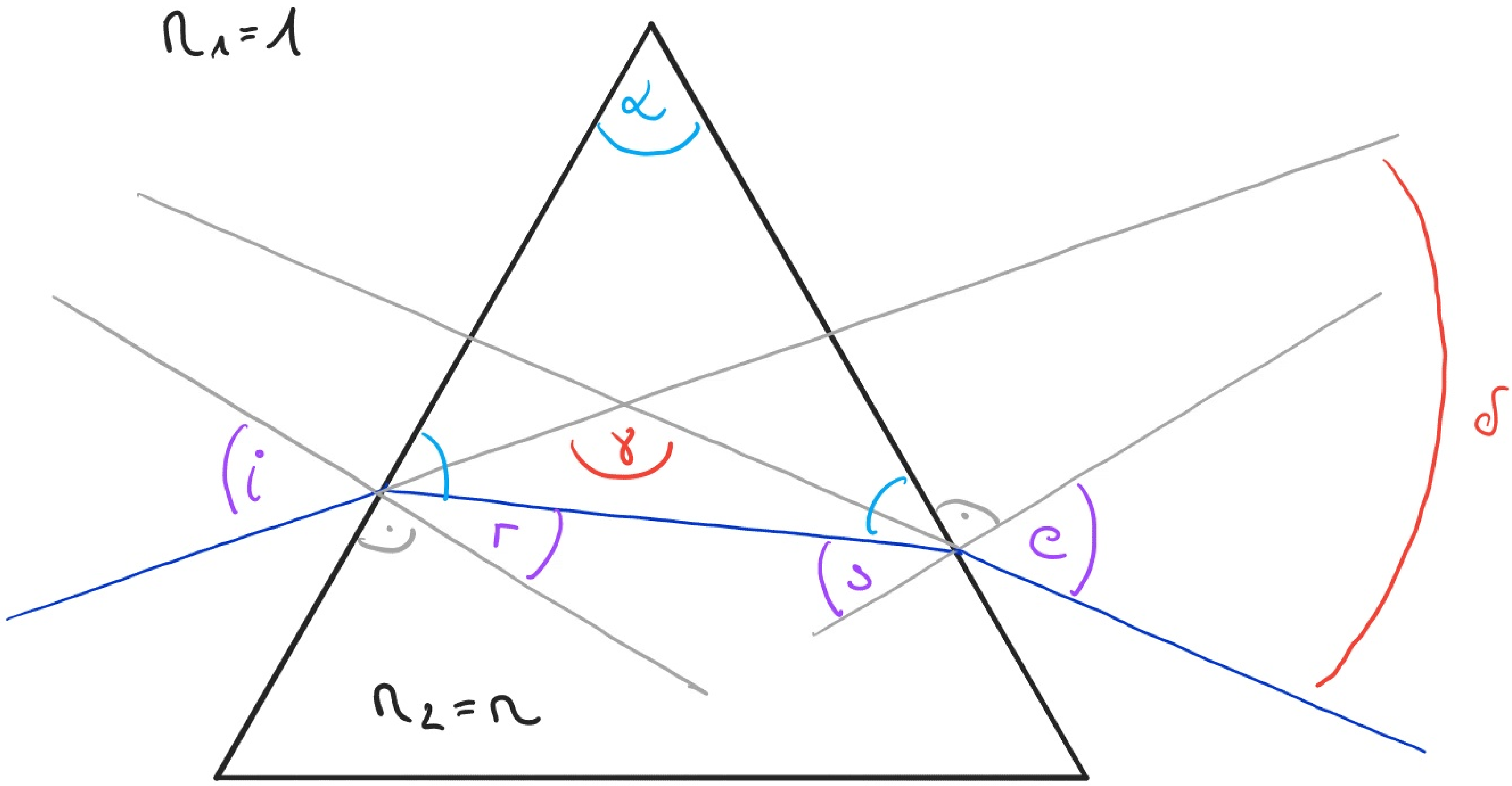
The picture introduced above is closely connected to Huygens' principle. It states that every point on a propagating wave front is the new starting point for an outgoing spherical, secondary wavelet (of the same frequency and speed). The interference of these wavelets then describes the propagation of the entire wave. This principle can be applied to plane as well as spherical wave fronts (see below). However, while being a useful tool to understand wave propagation, the principle does not correctly capture all interference aspects, i.e. it predicts backward propagation that is not observed.

the envelope of spherical wavelets captures the forward propagation of the entire wave front



5.) Example: Triangular prism

We would like to know by what angle δ is the light deflected upon exiting the prism. Consider the following geometry:



We first deduce that

$$\delta + \gamma = 180^\circ \quad \text{with} \quad \gamma = 180^\circ - (i-r) - (e-s)$$

$$\Rightarrow \quad \delta = 180^\circ - \gamma = i - r + e - s .$$

Summing the angles in the blue triangle, we find

$$180^\circ = \alpha + (90^\circ - r) + (90^\circ - s) \quad \Rightarrow \quad \alpha = r + s .$$

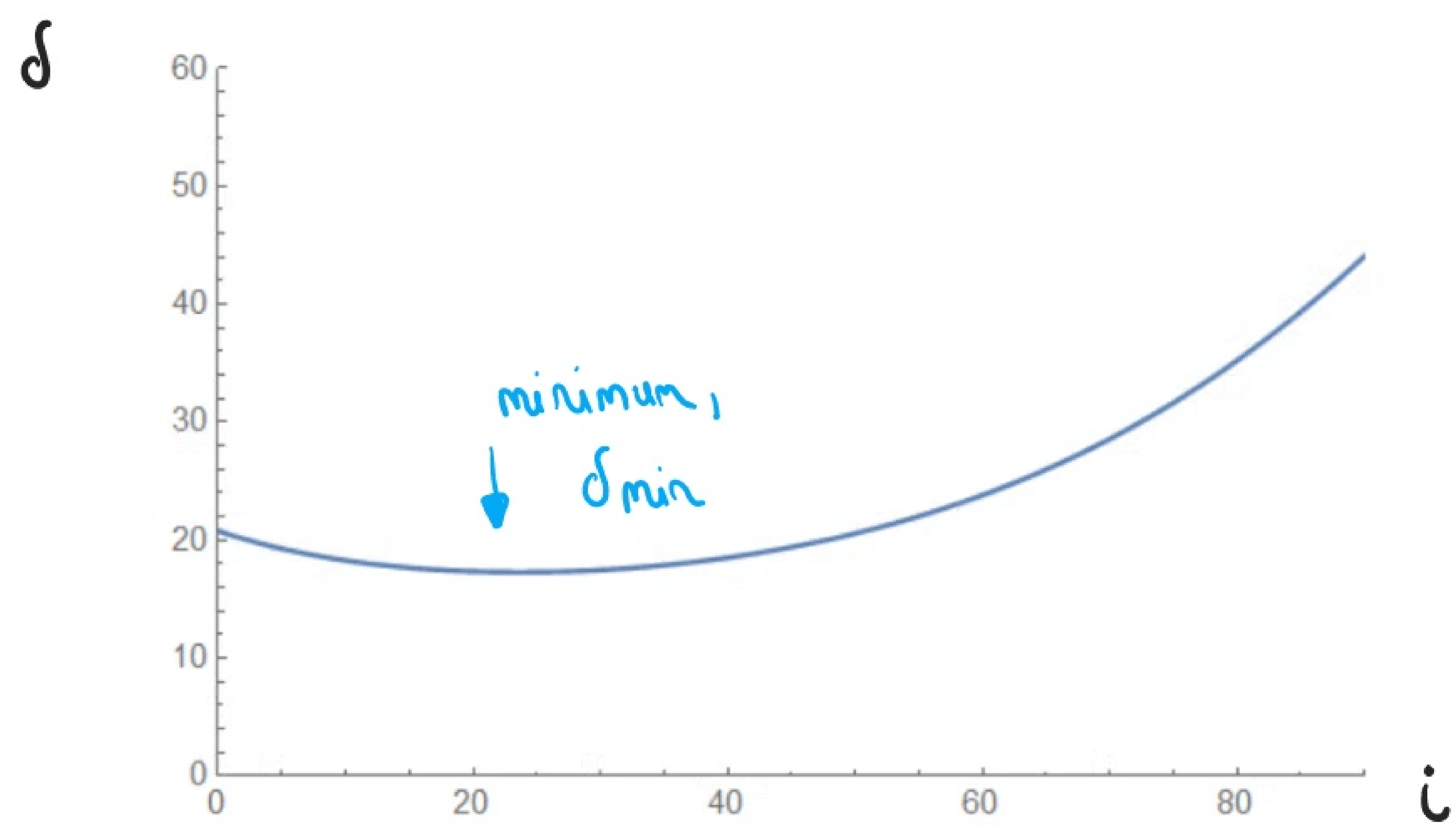
We are thus left with $\delta = i + e - \alpha$. Assuming that we know the angle α, we now want to express e in terms of the incident angle i and α. Applying Snell's law twice, we can find

$$
\begin{aligned}
e &= \mathrm{Arcsin}(n \sin s) = \mathrm{Arcsin}[n \sin(\alpha - r)] \\
&= \mathrm{Arcsin}(n \sin\alpha \cos r - n \cos\alpha \sin r) \\
&= \mathrm{Arcsin}\left(n \sin\alpha \sqrt{1 - \sin^2 i / n^2} - n \cos\alpha \sin i / n\right),
\end{aligned}
$$

$\sin r = \sin i / n$
$\cos r = \sqrt{1 - \sin^2 r}$

$$\Rightarrow \delta = i - \alpha + \mathrm{Arcsin}\left(n \sin\alpha \sqrt{1 - \sin^2 i / n^2} - n \cos\alpha \sin i / n\right).$$

We can plot this deflection angle for any given α as a function of the incidence angle i. For $\alpha = 30^\circ$, we obtain



The minimum deflection angle corresponds to the symmetric situation, where $e = i$ & $r = s$, and the light path in the medium is shortest (parallel to the prism's base). In this case

$$\delta = r = \alpha/2 \quad \text{and} \quad \delta_{min} = 2i - \alpha .$$

Measuring δ_{min} is useful as it allows us to deduce n:

$$n = \frac{\sin i}{\sin r} = \frac{\sin[(\delta_{min} + \alpha)/2]}{\sin(\alpha/2)} .$$

6.) Fermat's principle

There is an alternative perspective for the propagation of light, that has implications far beyond the field of Optics and is essential for many different areas of physics. It is closely connected to the variational principles you might remember from your classical mechanics course and is summarised in Fermat's law. It states that the optical path that light takes between two points is the path which minimises* the travel time of light. The travel time for a ray passing a physical distance l_j through a medium with n_j is equivalent to (for j different media)

$$T = \frac{1}{c} \sum_j n_j l_j .$$

* minimises is not completely correct; the path just has to be stationary (see below)

The optical path length is

$$OPL = \sum_j n_j l_j ,$$

or for a medium that has a continuously varying index of refraction through which light takes a path that can be parametrised by s :

$$OPL = \int_{P_1}^{P_2} n(s)\, ds .$$

The optical path length is a useful quantity as we can relate it to the number of wavelengths travelled by a wave (i.e. how many wave fronts fit in there), because for a distance l_j travelled in n_j, we have

$$\#\text{ of wavelengths} = \frac{l_j}{\lambda_t} = \frac{l_j n_j}{\lambda_0} = \frac{OPL}{\lambda_0}$$

using $\lambda_t = \frac{n_i}{n_t}\lambda_i$ where the incident material is vacuum

The OPL is thus the distance in vacuum that is equivalent to the distance the light traverses in a medium with n_j.

Alternatively, Fermat's principle can be restated as: Light takes the path that minimises the optical path length. More accurately, it should state that light takes the path that is stationary with respect to variations of this path. For a medium with continuously varying n this would correspond to

$$\delta \int_{P_1}^{P_2} n(s)\, ds \overset{!}{=} 0 .$$

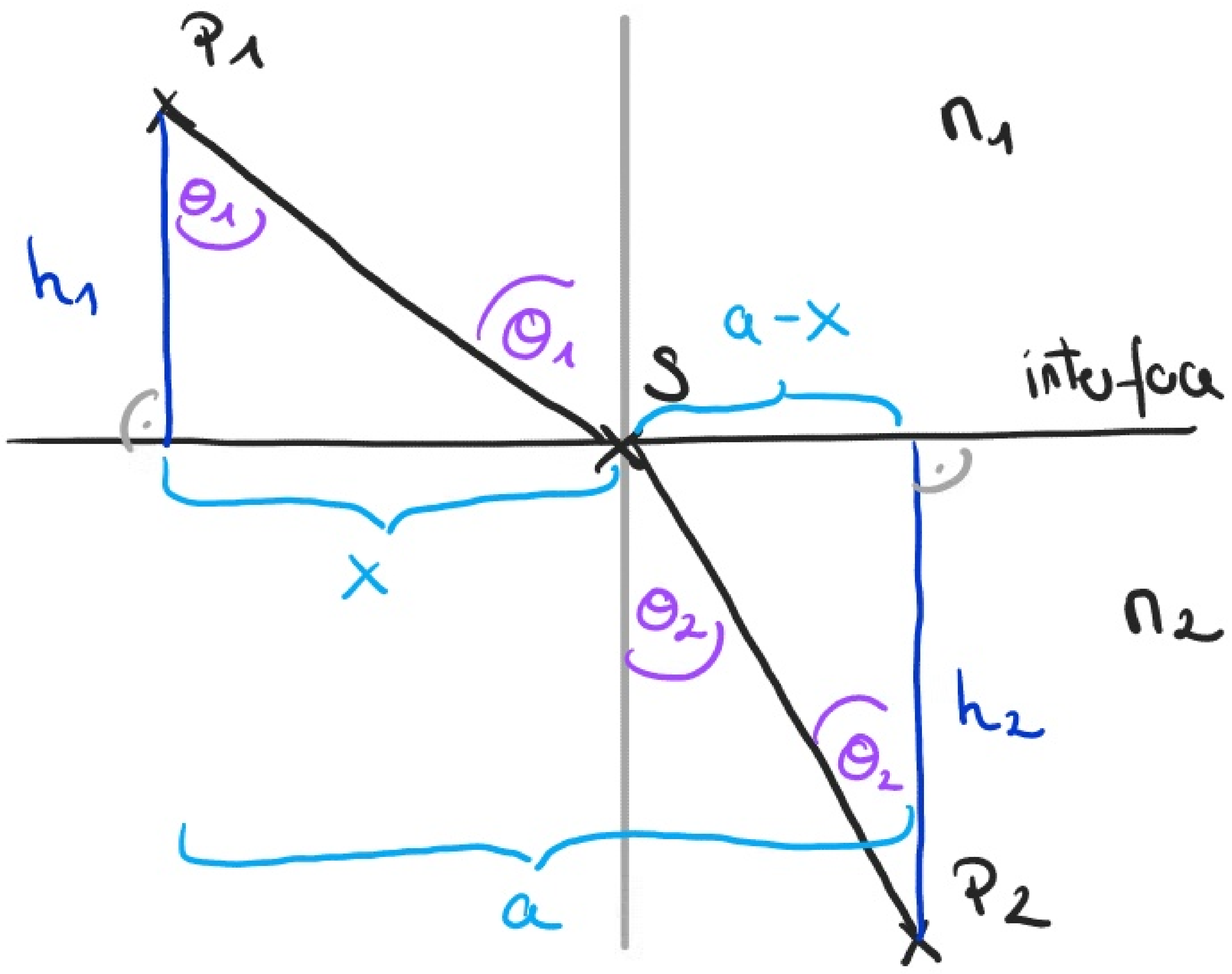
This is of the same form as e.g. the principle of least action (a varational principle applied commonly to mechanical systems), which reads

$$S[\vec{q}, t_1, t_2] = \int_{t_1}^{t_2} \mathcal{L}[\vec{q}(t), \dot{\vec{q}}(t), t]\, dt,$$

for an action S and a Lagrangian $\mathcal{L}$, that is a function of the generalised coordinates $\vec{q}$. Calculating $\delta S = 0$ provides information about the motion of particles. Similarly (although we will not explore this analogy further), $\delta OPL = 0$ provides the physically realised path of light.

7.) Example: Deriving Snell's law

Consider the following geometry (assuming that straight paths in constant-index media correspond to the shortest time):



The optical path length can change depending on the relative position of S with respect to P_1 and P_2. We can express the OPL in terms of the variable x:

$$OPL = \overline{P_1S} + \overline{SP_2} = n_1\sqrt{h_1^2 + x^2} + n_2\sqrt{h_2^2 + (a-x)^2}.$$

Minimising with respect to x yields:

$$\frac{\partial OPL}{\partial x} = \frac{n_1 x}{\sqrt{h_1^2 + x^2}} - \frac{n_2 (a-x)}{\sqrt{h_2^2 + (a-x)^2}} \stackrel{!}{=} 0 .$$

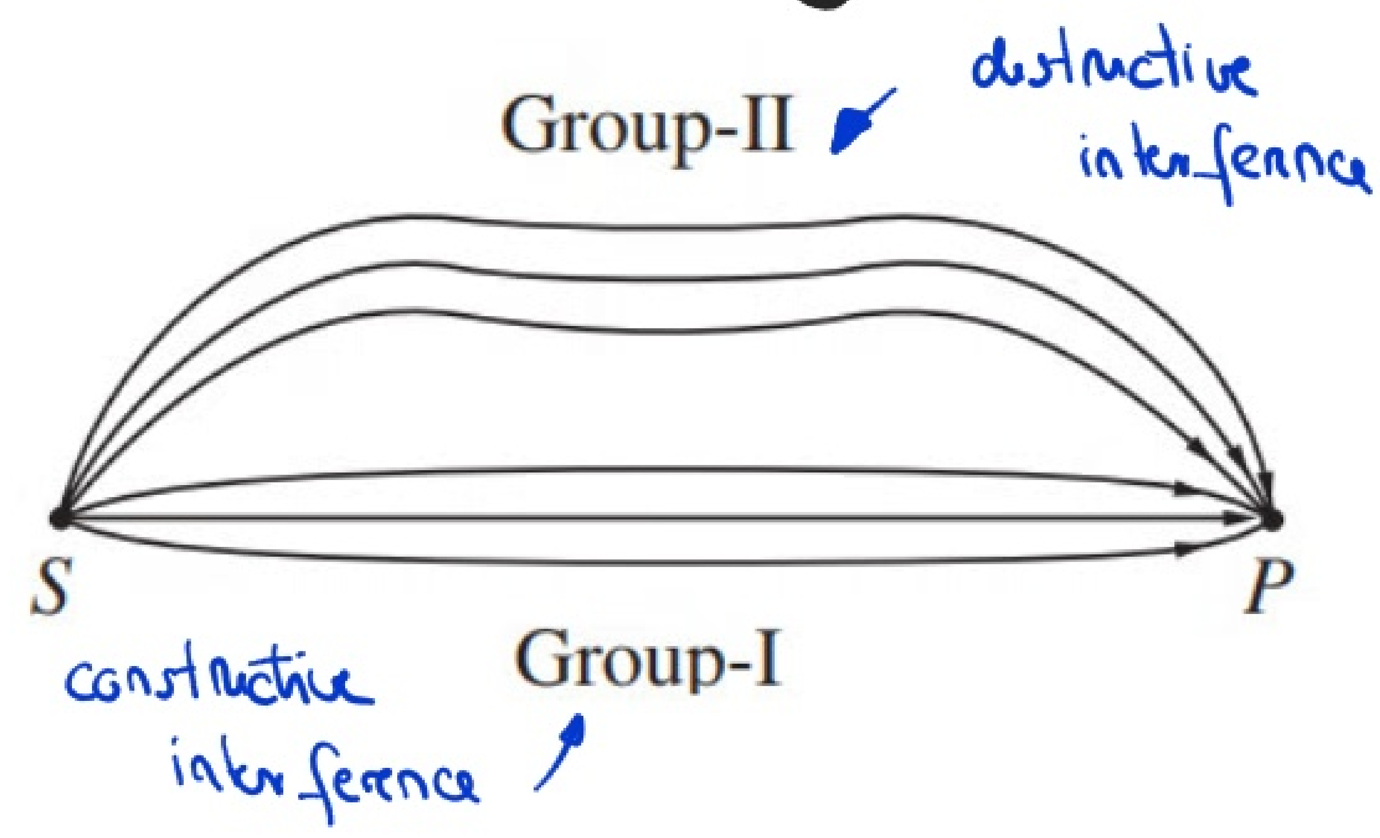
Looking at the geometry of the two right triangles, we find

$$\sin \Theta_1 = \frac{x}{\sqrt{h_1^2 + x^2}} \quad \text{and} \quad \sin \Theta_2 = \frac{a-x}{\sqrt{h_2^2 + (a-x)^2}} ,$$

$$\Rightarrow \quad \underline{\underline{n_1 \sin \Theta_1 = n_2 \sin \Theta_2}} . \qquad \blacksquare$$

8.) Connecting Fermat's & Huygens' principles

The idea of a stationary path can be understood from a microscopic perspective using Huygens' idea. According to the latter, each path connecting two points P_1 and P_2 is in principle possible (as each point generates outgoing spherical wavelets). However, only the one where wavelets interfere constructively is actually observed. This exactly corresponds to the stationary path. All other possible paths are dominated by destructive interference and hence not observed.



PHYS 434 – Lecture 4

EM Approach & Fresnel Equations

You should be familiar with those equations from your EM course, so we will not completely rederive them here. Instead they will be reviewed and their consequences discussed.

1.) Interface conditions

Using EM theory allows us to treat reflection and refraction at an interface in a much more quantitative way. We are able to not only address the direction of the wave, but also the polarisation-dependent intensity of the reflected and transmitted light.

Consider an EM field that propagates from a medium 1 with ε_1 and μ_1 into a medium 2 with ε_2 and μ_2. To determine how the electric and magnetic fields on either side of the interface are related, we can integrate Maxwell's equations over suitable paths along volumes that cross this interface.

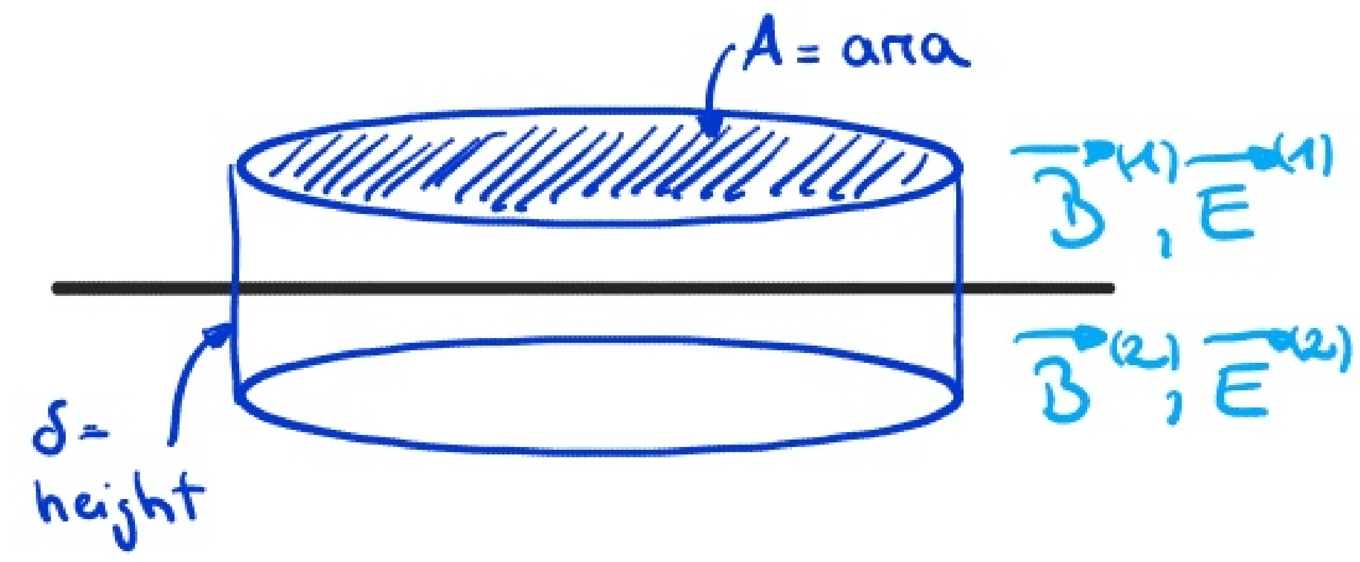
(i) For the Maxwell equations containing a divergence, we consider a pill box volume of height $\delta \to 0$ and area A. Integration leads to

divergence theorem / pointing along the surface normal

$$\int_{\text{pillbox}} (\nabla \cdot \vec{B})\, dV \overset{!}{=} \int_{\text{pillbox}} \vec{B} \cdot d\vec{A}$$

$$= B_\perp^{(1)} A - B_\perp^{(2)} A = 0$$

field components perpendicular to the interface

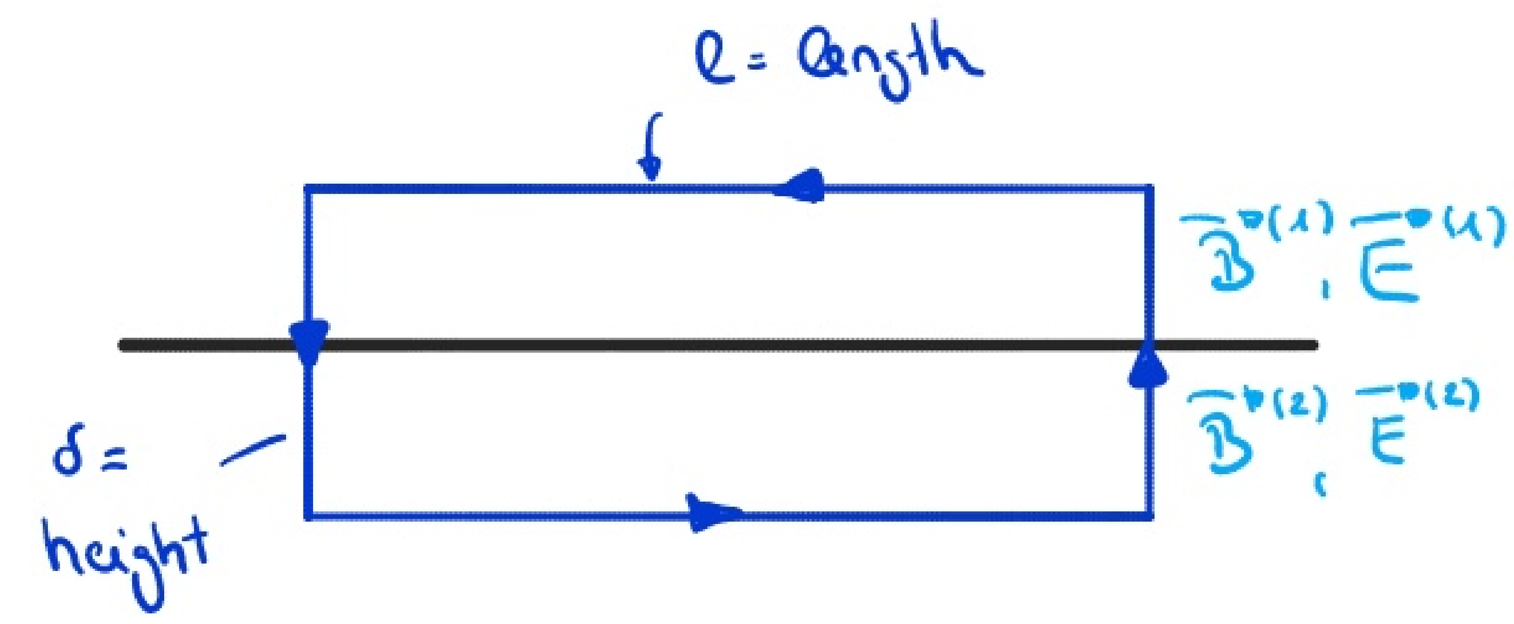


This suggests that the magnetic field components perpendicular to the surface have to be continuous; $B_\perp^{(1)} = B_\perp^{(2)}$. Similarly in the absence of free charges, we have $\nabla \cdot \vec{D} = 0$, which implies that $D_\perp$ is continuous and hence for the electric field components $\varepsilon_1 E_\perp^{(1)} = \varepsilon_2 E_\perp^{(2)}$.

(ii) For the Maxwell equations containing a curl, we consider a rectangular area with vanishing height $\delta \to 0$ and length l. Integration of Faraday's law then gives

Stokes' theorem / parallel to interface

$$\int_{\text{rectangle}} (\nabla \times \vec{E})\, d\vec{A} \overset{!}{=} \oint \vec{E}\, d\vec{l}$$



$$= E_\parallel^{(1)} l - E_\parallel^{(2)} l$$

This is equal to : $\displaystyle\int_{\text{rectangle}} -\frac{\partial \vec{B}}{\partial t}\, d\vec{A} \to 0$ for $\delta \to 0$

This suggests that $E_\parallel$ is continuous across the interface, $E_\parallel^{(1)} = E_\parallel^{(2)}$. In the absence of currents, we can deduce from Ampère's law $\nabla \times \vec{H} = \partial \vec{D}/\partial t$ that $H_\parallel$ is continuous across the interface or $B_\parallel^{(1)} \mu_2 = B_\parallel^{(2)} \mu_1$.

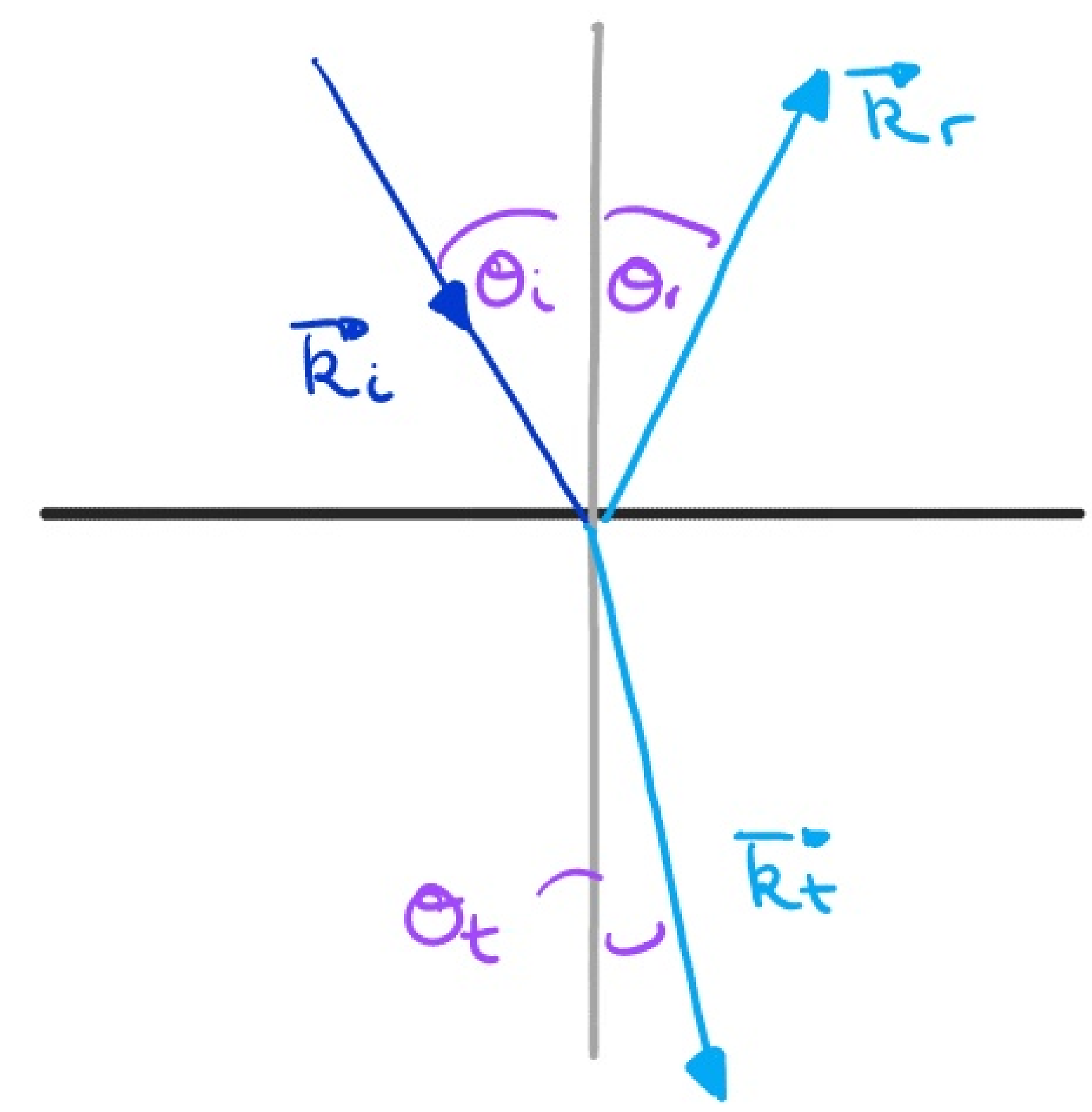
Note that if there are charges and currents present, we only have two interface conditions and know only that $D_\perp$ & $E_\parallel$ are continuous.

2.) Waves at an interface

Consider an incident plane wave of the form $\vec{E}_i = \vec{E}_{0i} e^{i(\vec{k}_i \vec{r} - \omega t)}$, where $\vec{E}_{0i}$ is constant in time and the wave thus linearly polarised. The reflected and transmitted electric fields must have the same functional time dependence as a result of the continuity of the electric field at the interface, which suggests that $\vec{E}_r = \vec{E}_{0r} e^{i(\vec{k}_r \vec{r} - \omega t)}$ and $\vec{E}_t = \vec{E}_{0t} e^{i(\vec{k}_t \vec{r} - \omega t)}$. This is again in agreement with the earlier observation that the oscillators in a medium emit radiation at the same frequency as the driving field. As a result of medium's response, the $\vec{E}_0$ will potentially contain a relative phase with respect to the incident wave.

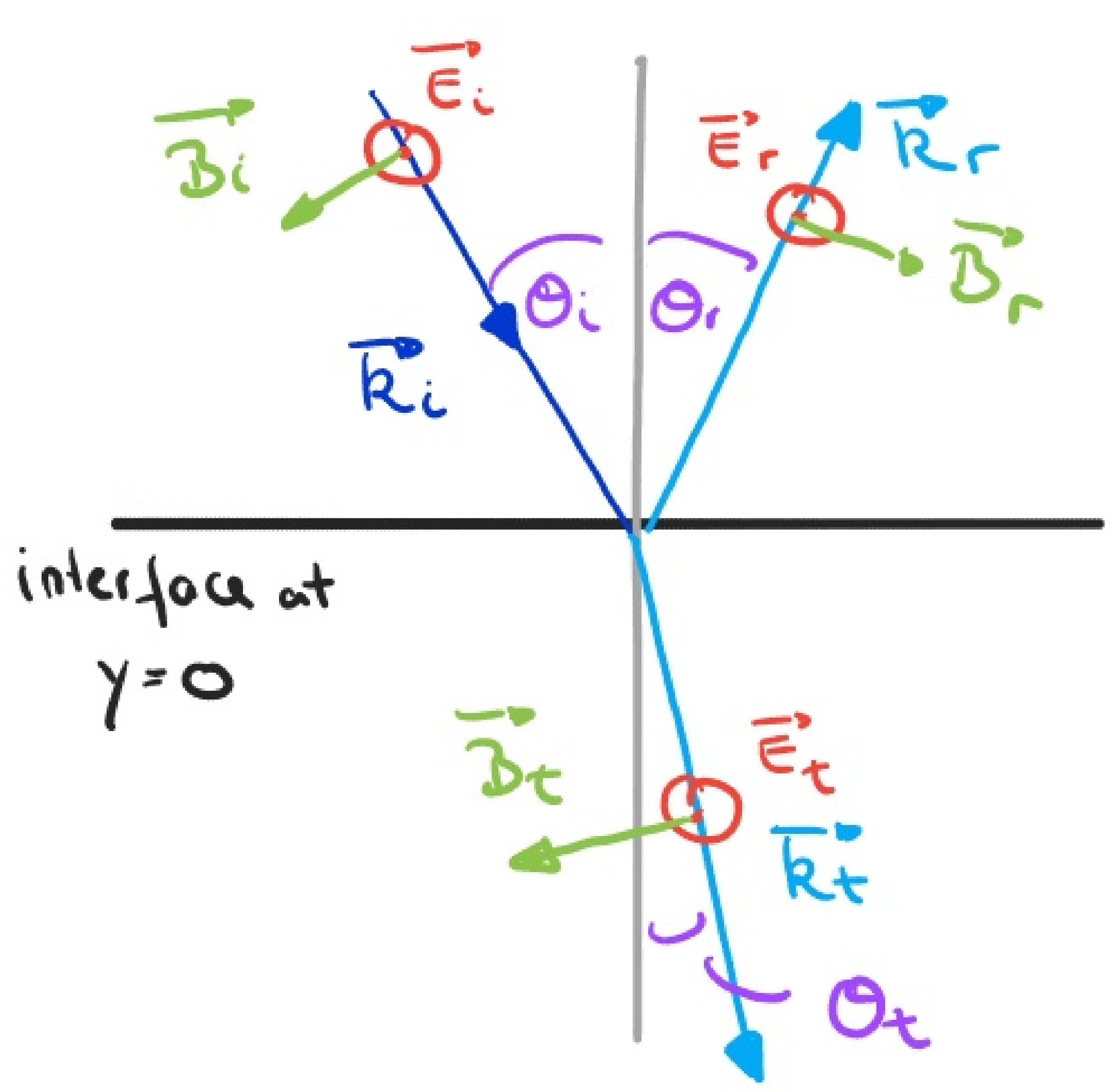
The wave vectors $\vec{k}_i$, $\vec{k}_r$ and $\vec{k}_t$ are now related by the law of reflection and Snell's law. We know that all three vectors will lie in one plane and using the following geometry, we can write

- $|\vec{k}_i| \equiv k_i = |\vec{k}_r| \equiv k_r$ ↲ wave equation
  $= \omega / v_1 = \omega\sqrt{\mu_1\varepsilon_1}$,
- $|\vec{k}_t| \equiv k_t = \omega / v_2 = \omega\sqrt{\mu_2\varepsilon_2}$,
- $\theta_r = \theta_i$,
- $k_i \sin\theta_i = k_t \sin\theta_t$.



Although the interface conditions provide relationships between the electric field amplitudes $\vec{E}_{0i}, \vec{E}_{0r}, \vec{E}_{0t}$, they are not uniquely determined so far. Their precise form depends on the polarisation of the incident light and we can distinguish two scenarios.

(i) $\vec{E} \perp, \vec{B} \parallel$ to the plane of incidence:

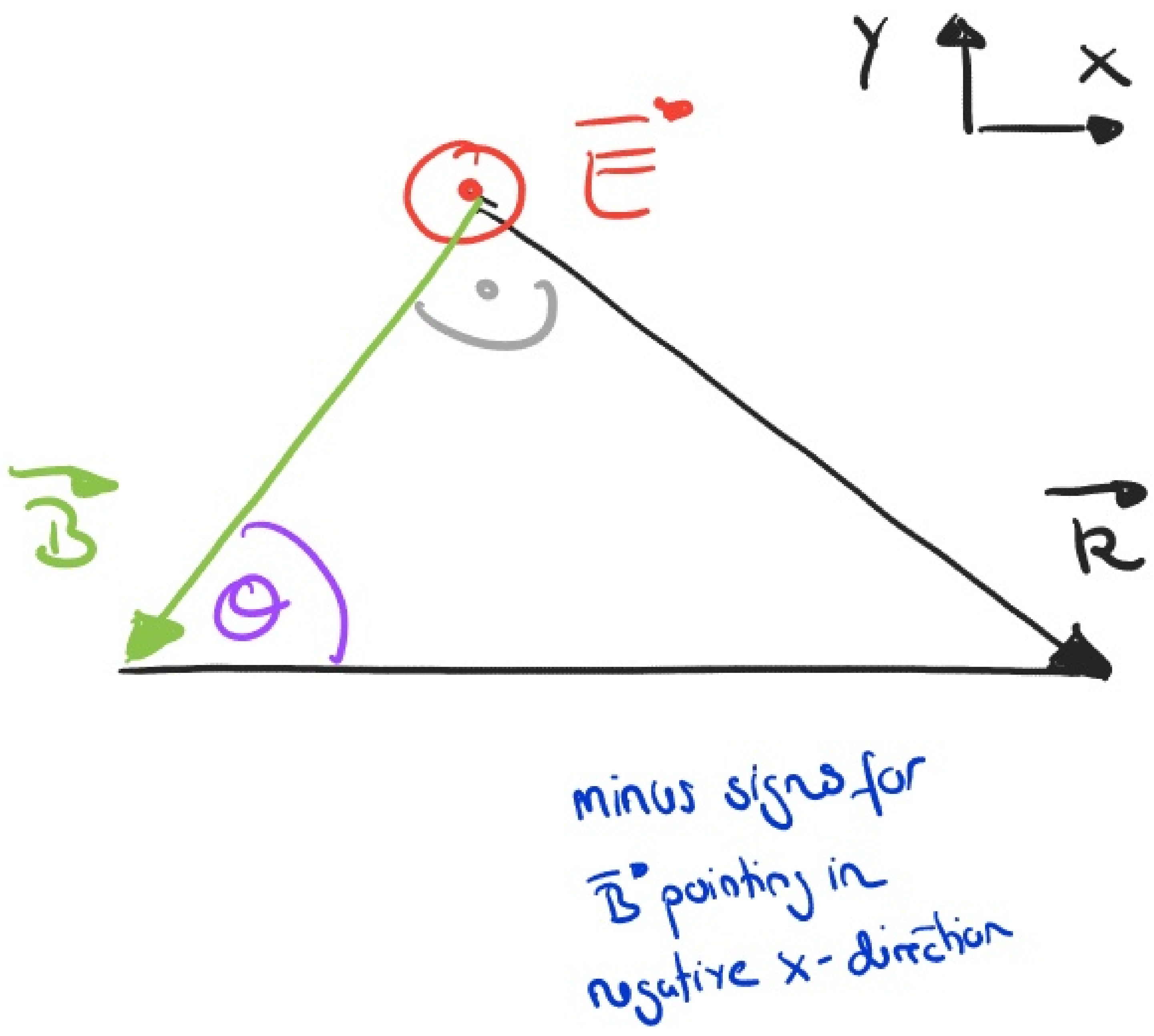


We choose (!) all polarisations / $\vec{E}$ fields to point out of the plane, in which case the electric field is entirely tangential to the interface. At the interface $y=0$, continuity thus dictates

$$\underline{\underline{E_{0i} + E_{0r} = E_{0t}}}. \quad (*)$$

The directions of the magnetic fields can be found from $\hat{k} \times \vec{E} = v\vec{B}$

and the right-hand rule for the cross product. From the geometry, we find for each of the three vectors



$$B_{\parallel} = B \cos\theta .$$

The continuity of the magnetic field $H_{\parallel} = B_{\parallel}/\mu$ thus gives another relationship:

$$-\frac{B_t}{\mu_2}\cos\theta_t = -\frac{B_i}{\mu_1}\cos\theta_i + \frac{B_r}{\mu_1}\cos\theta_r ,$$

$|B| = |E|/v$

$$\Rightarrow \frac{E_{ot}}{v_2\mu_2}\cos\theta_t = \frac{E_{oi}}{v_1\mu_1}\cos\theta_i - \frac{E_{or}}{v_1\mu_1}\cos\theta_r ,$$

$\theta_i = \theta_r$

$$\Rightarrow \frac{E_{ot}}{v_2\mu_2}\cos\theta_t = \frac{1}{v_1\mu_1}\cos\theta_i\,(E_{oi} - E_{or}) . \quad (**)$$

Equations (*) and (**) provide two equations for two unknowns, namely $r_\perp \equiv E_{or}/E_{oi}$ and $t_\perp \equiv E_{ot}/E_{oi}$. Remembering that $v = c/n$, we can derive the following two Fresnel equations for any linear, isotropic and homogeneous medium. Typically we have $\mu_0 = \mu_1 = \mu_2$ and thus

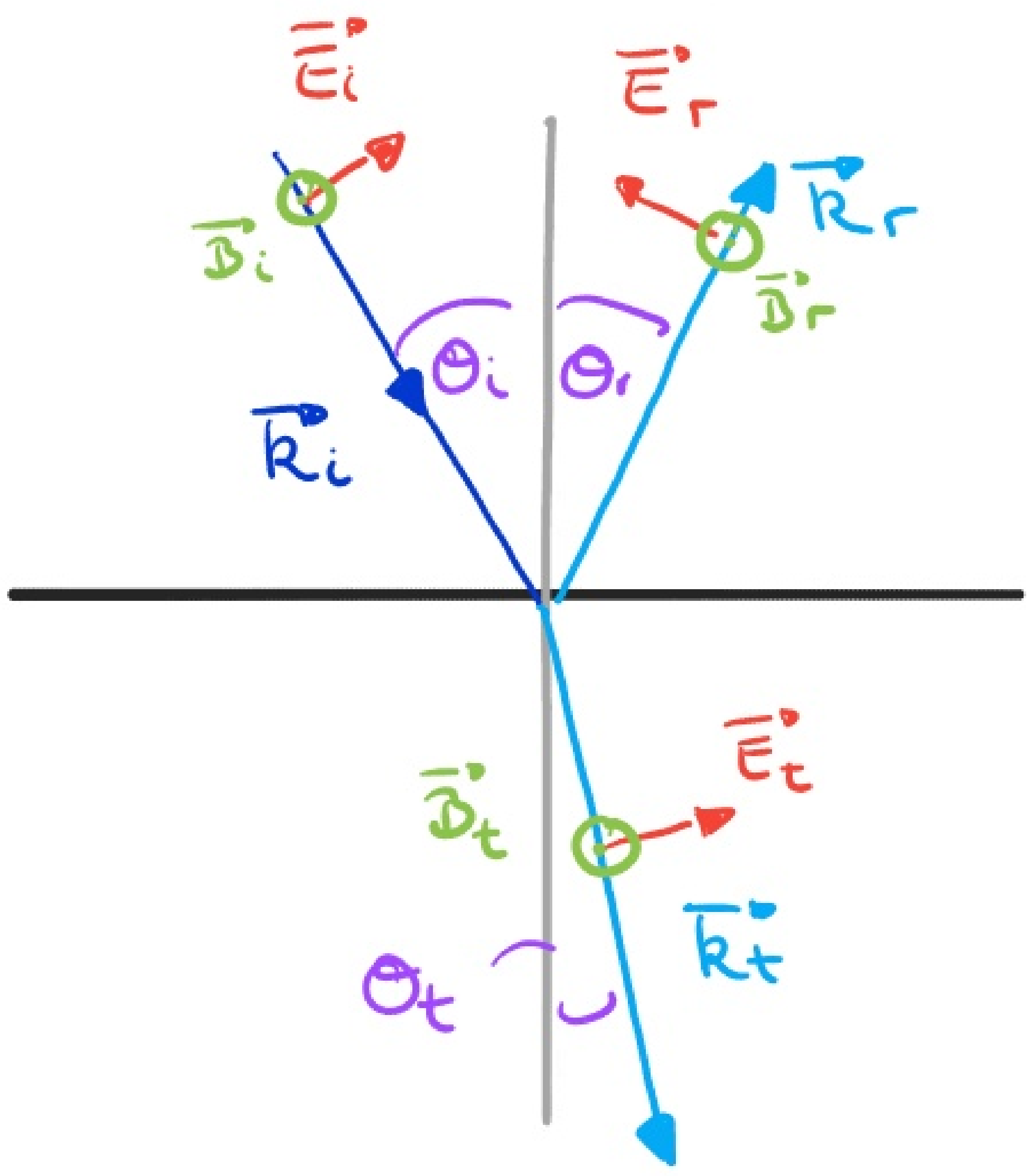
$$r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} ,$$

amplitude reflection coefficient

$$t_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} .$$

amplitude transmission coefficient

(ii) $\vec{E} \parallel, \vec{B} \perp$ to the plane of incidence :



We take the same approach as before but account for the changes in the interface conditions. Choosing all polarisations in the plane of the paper, continuity of the tangential components of $\vec{E}$ leads to

$$E_{0t} \cos\theta_t = E_{0i} \cos\theta_i - E_{0r} \cos\theta_r .$$

The magnetic field components are now pointing out of the paper plane and the continuity of the tangential fields suggests

$$\frac{1}{\mu_2} B_t = \frac{1}{\mu_1}(B_i + B_r) \Rightarrow \frac{1}{v_1 \mu_1}(E_{0i} + E_{0r}) = \frac{1}{v_2 \mu_2} E_{0t} .$$

For $\mu_1 = \mu_2 = \mu_0$, we arrive at a second set of Fresnel equations for $r_\parallel = E_{0r} / E_{0i}$ and $t_\parallel = E_{0t} / E_{0i}$, namely

$$r_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i} ,$$

amplitude reflection coefficient

$$t_{\parallel} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} .$$

amplitude transmission coefficient

Note that the precise form of the Fresnel equations depends on the directions of the respective vectors (that we choose somewhat arbitrarily). They therefore refer to amplitudes with a specific sign convention, that is illustrated in the sketches above. You might thus find Fresnel equations in the literature that have different signs.

3.) Interpretation

We can now use these equations to study the fractional amplitudes and flux densities that are reflected / refracted as well as possible phase shifts. We distinguish again two cases.

(i) External reflection, $n_t > n_i$:

- $\theta_i = 0$ (normal incidence): $\theta_t = 0$

$$r_{\perp} = \frac{n_i - n_t}{n_i + n_t} = - r_{\parallel} \quad \text{and} \quad t_{\perp} = \frac{2 n_i}{n_i + n_t} = t_{\parallel} ,$$

→ $r_{\perp} < 0$ implies a phase shift of π for the reflected beam; the larger the index contrast the larger $r_{\parallel}, r_{\perp}$.

- $\theta_i \to 90°$ (glancing incidence):

$$r_\perp = r_\parallel \longrightarrow -1 \quad \text{and} \quad t_\perp = 0 = t_\parallel,$$

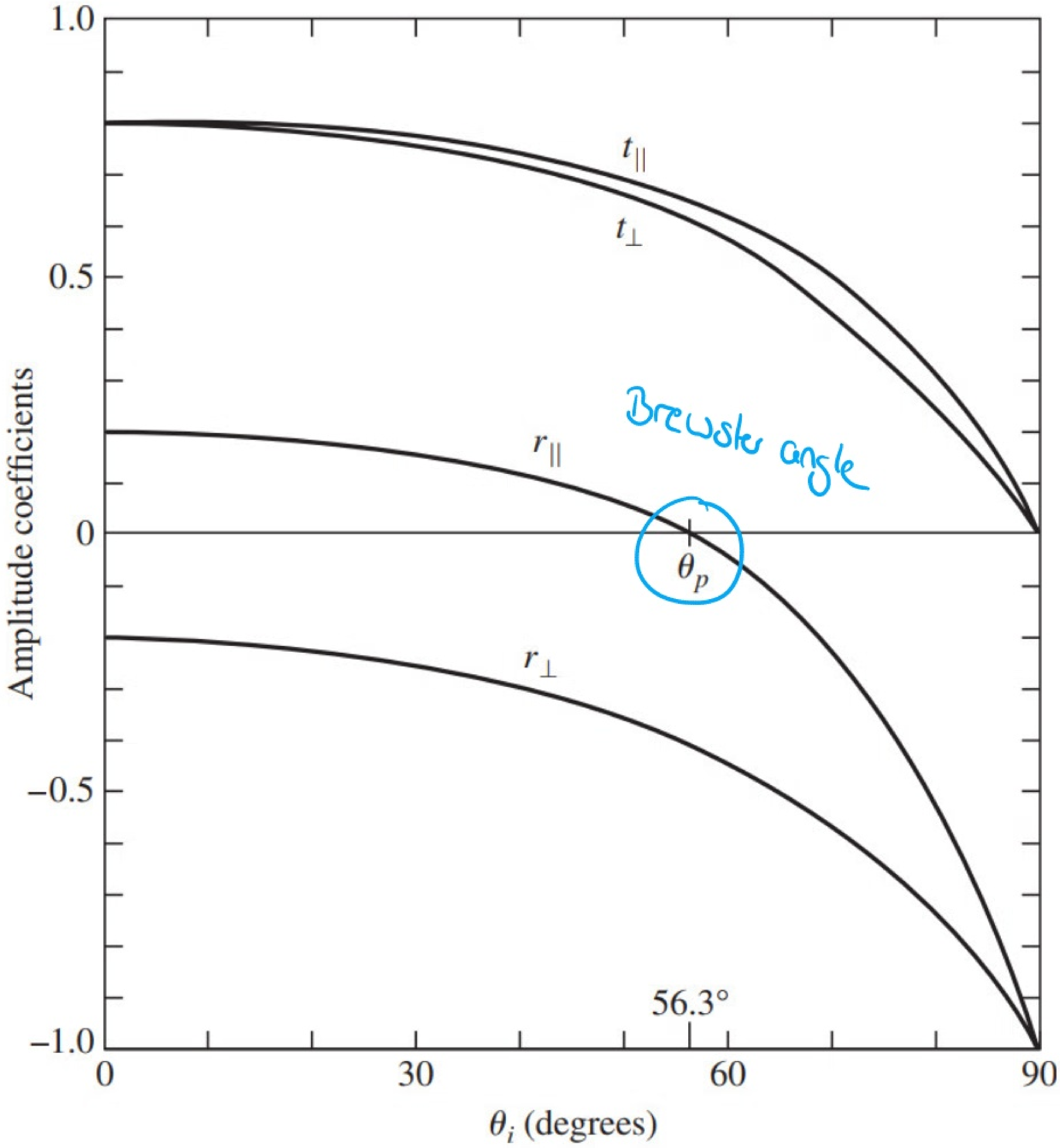
$\Rightarrow$ $r_\perp$ and $r_\parallel < 0$ corresponding to a π phase shift; everything is reflected and nothing transmitted, which suggest that the interface acts like a mirror.

- $r_\parallel$ starts out positive but ends being negative; there is some angle $\theta_i = \theta_p$ at which $r_\parallel = 0$. At this so-called Brewster angle, $t_\parallel = 1$, i.e. the medium is perfectly transmittant for $\parallel$ polarisation. If unpolarised light hits the interface, the reflected beam will be $\perp$ polarised. θ_p is thus also called the polarisation angle.

The full behaviour of these coefficients as a function of θ_i is illustrated for an air-glass interface on the next page.

(ii) Internal reflection, $n_i > n_t$:

- In this case, $\theta_t > \theta_i$ and thus $r_\perp$ always positive, while $r_\parallel$ changes its sign. At the angle θ_p' we thus have again $r_\parallel = 0$. This polarisation angle

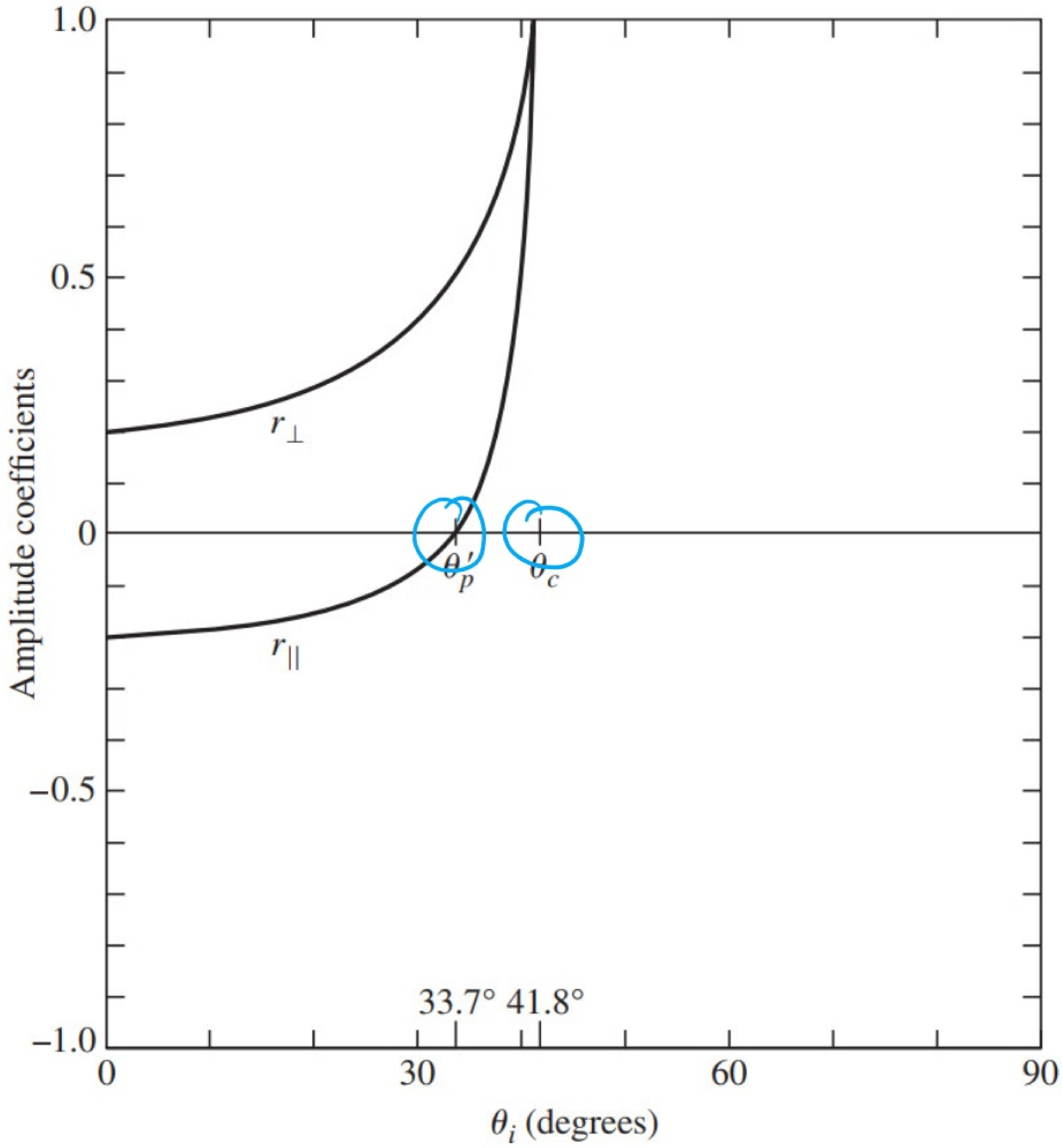


external reflection for air-glass interface, $n_i = 1$, $n_t = 1.5$

is complementary to θ_p and it can be shown that $\theta_p + \theta_p' = 90°$.

- There is also a maximum incidence angle θ_c at which both reflection amplitude coefficients go to 1 and $t_{\parallel} = t_{\perp} = 0$. This situation is referred to as 'total internal reflection'.

The full behaviour of these coefficients as a function of θ_i is illustrated for a glass-air interface on the next page.



internal reflection for glass-air interface

$n_i = 1.5$

$n_t = 1$

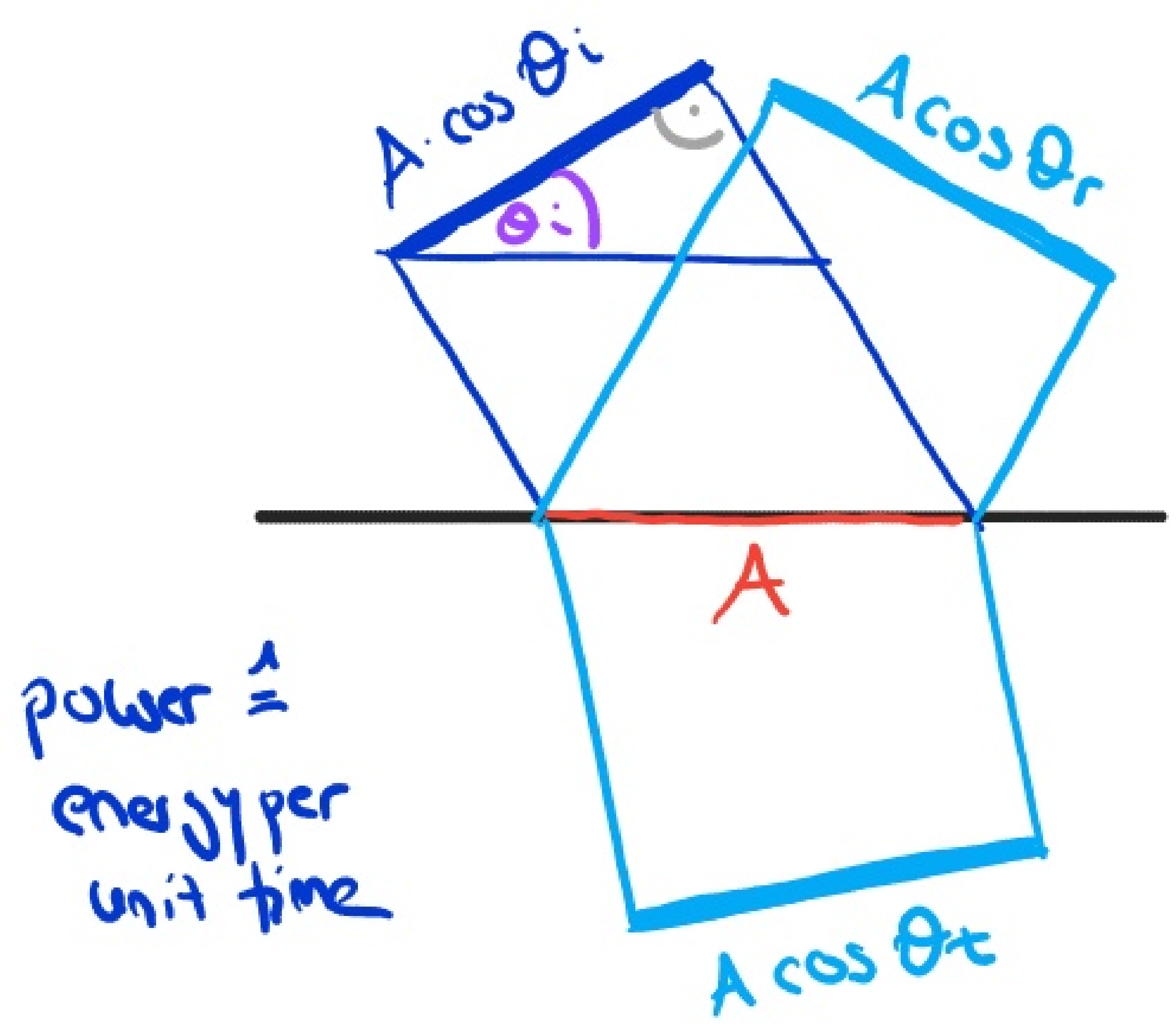
In both reflection scenarios, we can show that

$$t_\perp + (-r_\perp) = 0 \quad \text{and} \quad t_\parallel + r_\parallel = 1 .$$

Finally note that the reflection coefficients for internal and external reflection have indeed opposite signs for $\theta_i = 0$. This corresponds to a 180° phase shift between the electric fields, which we already concluded from our thought experiment at the beginning of Lecture 3.

4.) Reflectance and transmittance

If we consider light as a circular beam that illuminates an area A on the interface, we want to know the energy per time stored in the incident, reflected and transmitted beam. Denoting the flux densities as I_i, I_r and I_t, respectively, and the cross sectional areas as $A\cos\theta$ (see sketch), we can deduce the power $I \cdot A\cos\theta$, in each respective beam. We can now use these to define the reflectance and transmittance of a medium as



$$\theta_r = \theta_i \qquad I = \langle S \rangle_T = \frac{c\varepsilon_0}{2} E_0^2$$

$$\underline{\underline{R}} \equiv \frac{I_r \cancel{A\cos\theta_r}}{I_i \cancel{A\cos\theta_i}} = \frac{I_r}{I_i} = \frac{\cancel{v_1}\,\cancel{\varepsilon_1}\, E_{0r}^2/2}{\cancel{v_1}\,\cancel{\varepsilon_1}\, E_{0i}^2/2} = \underline{\underline{r^2}},$$

$$\underline{\underline{T}} = \frac{I_t \cancel{A}\cos\theta_t}{I_i \cancel{A}\cos\theta_i} = \frac{\cos\theta_t\, v_2\, \varepsilon_2\, E_{0t}^2}{\cos\theta_i\, v_1\, \varepsilon_1\, E_{0i}^2} = \underline{\underline{\frac{n_t\cos\theta_t}{n_i\cos\theta_i}\, t^2}}.$$

$$n = c/v = \sqrt{\varepsilon/\varepsilon_0} \quad \text{for } \mu = \mu_0$$

Specifically note that $T \neq t^2$ but dependent on the angles and refractive indices of the two media, because the wave speed changes across the interface and the crosssectional areas of the two beams differ.

Considering conservation of energy of the entire configuration, it is possible to show that

$$\underline{\underline{R_\perp + T_\perp = 1}} \quad \text{and} \quad \underline{\underline{R_\parallel + T_\parallel = 1}},$$

as one would expect in the absence of absorption.

Finally, for normal incidence, where we cannot distinguish between the different polarisations

$$R = R_\perp = R_\parallel = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2 \quad \text{and} \quad T = T_\perp = T_\parallel = \frac{4 n_t n_i}{(n_t + n_i)^2} .$$

This leads to a value of about 4% being reflected, if we are considering an air-glass interface.

5.) Total internal reflection

We already saw that for internal reflection, at some critical angle θ_c the reflected amplitude coefficient r approaches 1. The critical angle is that value of θ_i, where $\theta_t = 90°$. In this case, Snell's law gives $\sin\theta_i = \sin\theta_c = n_t/n_i$.

Nevertheless, to satisfy the interface conditions, there must be an an electromagnetic field on the other side of the interface. This field however does not carry any energy away as $T_\perp$ and $T_\parallel \to 0$ for $\theta_i = \theta_c$. Instead, what happens is that above the critical angle, the transmitted wave vector becomes complex. We can see this by considering a transmitted wave of the form $\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t\vec{r} - \omega t)}$. In our 2D geometry we have $\vec{k}_t = (k_t \sin\theta_t,\ k_t \cos\theta_t)$. Using Snell's law we find for angles $\theta_i > \theta_c$ (remember $\theta_t = 90°$) that $\sin\theta_i > n_t/n_i$ and thus

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\sin^2\theta_i}{\underbrace{(n_t/n_i)^2}_{>1}}}$$

$$= i\sqrt{\frac{\sin^2\theta_i}{(n_t/n_i)^2} - 1} \equiv i\alpha$$

Substituting this into $\vec{k}_t$ and subsequently $\vec{E}_t$ gives a dependence $\propto e^{-\beta y}$. The transmitted wave decays exponentially into the medium and is thus also referred to as the 'evanescent' wave. The quantity β is referred to as the attenuation coefficient and given by $\beta = k_t \alpha$.