Master UAB - *High Energy Physics, Astrophysics and Cosmology*

NSs, BHs and GWs

<u>TOWARDS GENERAL RELATIVITY</u>

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Literature suggestions:

- *A First Course in General Relativity*, B. Schutz
- *Black Holes and Time Warps*, K. Thorne
- *Introducing Einstein's Relativity*, R. D'Inverno
- *Gravitation*, C. Misner, K. Thorne & J. Wheeler
- *General Relativity*, R. Wald

- *Lecture Notes on General Relativity*, S. M. Carroll, https://arxiv.org/pdf/gr-qc/9712019.pdf
- *Geometry and physics of BH*, É. Gourgoulhon, lecture notes, https://luth.obspm.fr/~luthier/gourgoulhon/bh16/

Overview:

Covered so far: NS EoS, transport properties, NS timing, magneto-thermal evolution

1. **Special relativity**
2. **Tensor calculus**

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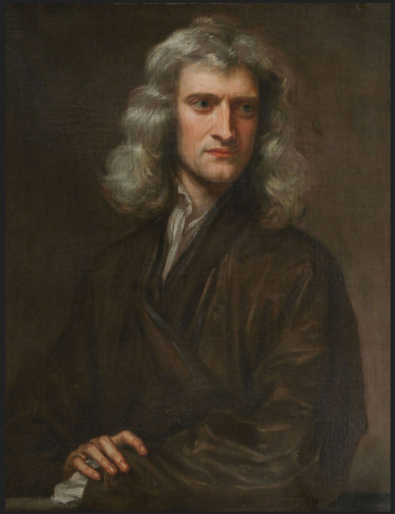
Covered so far: NS EoS, transport properties, NS timing, magneto-thermal evolution

1. **Special relativity (SR)**
2. Tensor calculus

1.1 SR - Newtonian gravity

- Until the 19th century, **Newton's theory of gravity** had been successfully applied to many phenomena, e.g., Earth & Moon.

$$F = G\frac{m_1 m_2}{r^2}$$



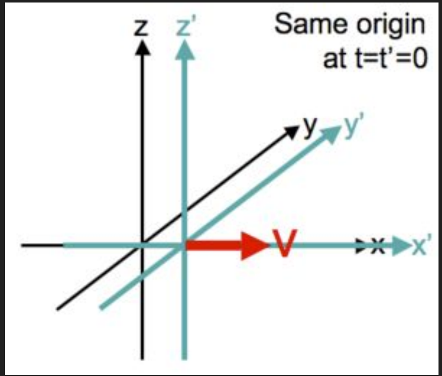
- He considered **space** & **time** as **absolute**: space exists independently of bodies within it; time exists without anyone measuring it.
- Their existence does not depend on physical events and the quantities are **distinct**.

1.2 SR - Galilean transformations

- In the Newtonian picture, an experimenting **observer** (clock + ruler) will recover the same laws of physics, if their local frame of reference is an **inertial frame** (no net forces acting).

- Inertial frames are at rest or travel with constant velocity relative to each other. They are connected via **Galilean transformations**:

$$x' = x - vt, \qquad y' = y, \qquad z' = z, \qquad t' = t$$



- **Velocities** can be treated **like vectors**.

1.3 SR - Maxwell's electromagnetism

- By 1870, Maxwell published a **classical theory of electromagnetism**: electricity, magnetism & light are manifestations of one phenomenon.

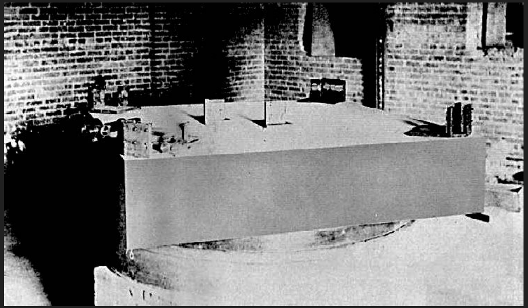
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

- These show that EM waves travel with (constant) **speed,** c.
- Wave phenomena were known to require propagation media, which was postulated as all-pervading **'luminiferous aether**'.

1.4 SR - incompatibility

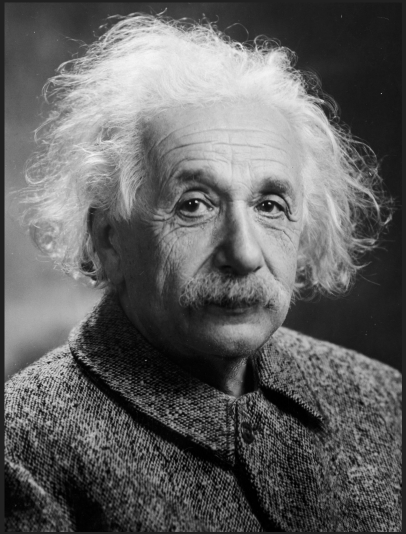
- While Newtonian laws are Galilean invariant, **Maxwell's equations** are **not**. Put differently, the speed of light is not additive and cannot depend on the source/observer velocity.
- In 1887, the **Michelson-Morley experiment** (measuring the speed of light in two arms of an interferometre) suggested that the aether did not exist.



- To reconcile these issues, Einstein took a **new approach** at combining earlier results (in particular by Lorentz and Poincaré) and published a **Special Theory of Relativity**.

1.5 Postulates of SR

- Einstein started by assuming that the following two postulates are valid without restrictions:
    - **Principle of relativity:** laws of physics are identical in inertial frames; all inertial frames are equivalent.
    - **Constancy of c**: speed of light in vacuum is the same for all observers, independent of the relative motion of the source.



- This implies that **Newtonian gravity cannot** always be **valid**. It has to be adjusted for large velocities or strong gravity.

1.6 SR - Lorentz transformations

- The SR postulates are equivalent to the statement that the laws of physics are **invariant** under **Lorentz transformations**.
- For an **event** with coordinates (t, x) in a reference frame S, and coordinates (t’, x’) in S’ moving with v relative to S, we have

$$t' = \gamma(t - vx/c^2), \qquad x' = \gamma(x - vt)$$

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

ɣ is the so-called **Lorentz factor**. The inverse transformation:

$$t = \gamma(t' + vx'/c^2), \qquad x = \gamma(x' + vt')$$

1.7 SR - a four-dimensional world

- Whereas absolute time in Newtonian physics remains invariant under a Galilean transformation, **Lorentz transformations mix space and time**. The two are no longer separate.
- In special relativity, time and space merge into a four-dimensional continuum (a so-called manifold) named **spacetime**.
- The **interval** between two events (t, x, y, z) and (t+dt, x+dx, y+dy, z+dz) that is invariant under Lorentz transformations is

$$ds^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

Minkowski spacetime

1.8 Questions

- Go to www.menti.com & enter 3821 5622.
  - 1. In a Newtonian picture can velocities be added and subtracted like vectors?
    - Yes
    - No
  - 2. Which of the following no longer holds universally in SR?
    - Principle of relativity
    - Newtonian gravity
    - Constancy of speed of light

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    - Newtonian gravity
    - Constancy of speed of light

1.9 Exercise

- Show that ds^2 in one spatial dimension is invariant under the (infinitesimal) Lorentz transformation, i.e., ds^2 is independent of the frame it is measured in. Use the following:

$$ds^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$$

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

$$\mathrm{d}t = \gamma(\mathrm{d}t' + v\mathrm{d}x'/c^2), \qquad \mathrm{d}x = \gamma(\mathrm{d}x' + v\mathrm{d}t')$$

1.9 Exercise

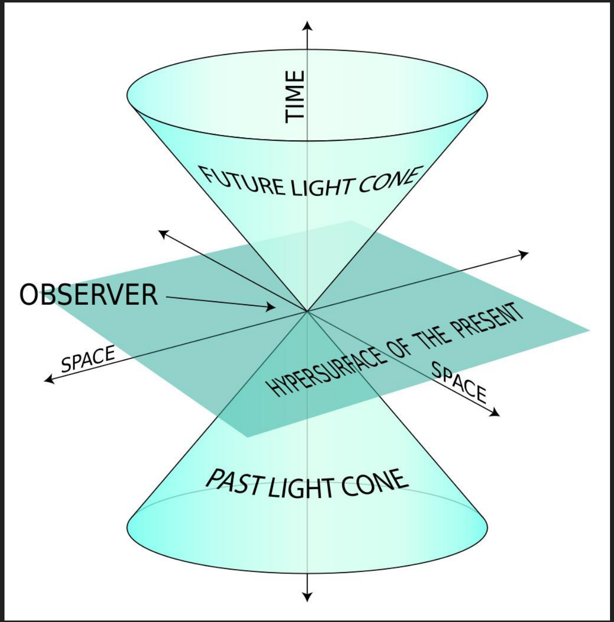
- Answer:

$$
\begin{aligned}
c^2 \mathrm{d}t^2 - \mathrm{d}x^2 &= c^2\gamma^2 \left(\mathrm{d}t' + v\mathrm{d}x'/c^2\right)^2 - \gamma^2(\mathrm{d}x' + v\mathrm{d}t')^2 \\
&= \gamma^2 \left(c^2\mathrm{d}t'^2 + v^2\mathrm{d}x'^2/c^2 - \mathrm{d}x'^2 - v^2\mathrm{d}t'^2\right) \\
&= \gamma^2 c^2 \mathrm{d}t'^2(1 - v^2/c^2) - \gamma^2 \mathrm{d}x'^2(1 - v^2/c^2) \\
&= c^2\mathrm{d}t'^2 - \mathrm{d}x'^2
\end{aligned}
$$

1.10 SR - light cone

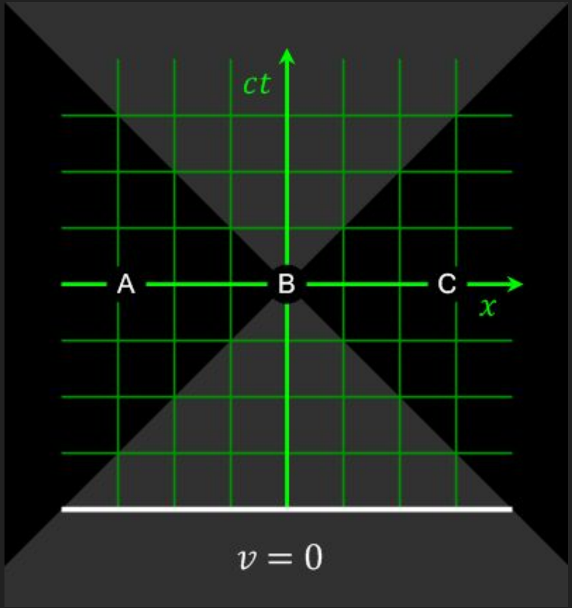
$$ds^2 = c^2 dt^2 - dx^2$$

- We can distinguish 3 different cases:
  - $ds^2>0$: two events separated by more time than space - **timelike.**
  - $ds^2<0$: two events separated by more space than time - **spacelike.**
  - $ds^2=0$: two events are **lightlike** separated, and $|dx/dt|=c$.
- These are used to construct **spacetime diagrams / light cones** to illustrate past, future, elsewhere & causality.



1.11 SR - simultaneous or not?

- Because c is constant all inertial observers will construct the **same light cone**. Past and future events keep that property.



- The situation is different for spacelike separated events: consider three events (A, B, C) that happen **simultaneously** for an observer with v=0.

- An observer with v≠0 will disagree. For v=0.3c, the events happen in the order (C, B, A).

$$dt = \gamma(dt' + v\,dx'/c^2)$$

1.12 SR - measuring time & length

- Consider a **clock at rest** in a rest frame S that measures two events (t, x) and (t+T, x). In a frame S' moving with v, we find

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = \gamma \Delta t \quad \Rightarrow \quad T' = \gamma T$$

  - **Time dilation**: time between events is not invariant but depends on the observers' speed. <u>Moving clocks go slower</u>!

- Consider a ruler of length Δx at rest and aligned along x in S. Measuring its length in S' gives

$$\Delta x' = \Delta x/\gamma - v\Delta t' = \Delta x/\gamma$$

  - **Length contraction**: <u>Moving rulers are shorter!</u>

1.13 SR - Doppler effect (DE)

- The classical Doppler effect arises when the source or receiver move relative to each other and in a medium. In SR, we don't need the medium as a reference but account for **time dilation**.

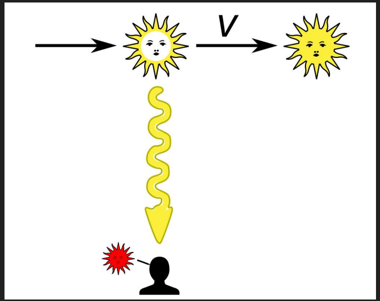
- In the source frame S for the receiver moving away with v>0:

$$\lambda_{\text{rec}} = \sqrt{(1 + v/c)/(1 - v/c)}\, \lambda_{\text{source}}$$

Longitudinal DE

- SR also predicts a **transverse DE**. If the receiver sees the source at its closest point:

$$\lambda_{\text{rec}} = \gamma \lambda_{\text{source}}$$



1.14 SR - relativistic mechanics

- Based on what we now know, Newtonian mechanics can be modified to incorporate objects moving with speeds close to c:
  - **Relativistic mass** differs from rest mass: $m(\mathbf{v}) = \gamma m_0$
  - **Energy** and **momentum** are also adjusted and obey a relativistic energy-momentum relation:

$$E = mc^2, \quad \mathbf{p} = m\mathbf{v} \quad \Rightarrow \quad E = \sqrt{m_0^2 c^4 + p^2 c^2}$$

  - For (quantised) **photons**, we have: $E = hc/\lambda, \quad p = h/\lambda$

1.15 Questions

- Go to www.menti.com & enter 1341 6537.
  - 3. When the separation between two events is given by $ds^2<0$, they are characterised by what?
    - Lightlike separation
    - Timelike separation
    - Spacelike separation
  - 4. Is there a classical transverse Doppler effect?
    - Yes
    - No

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Overview:

Covered so far: special relativity

1. Special relativity
2. **Tensor calculus (TC)**

2.1 TC - manifolds

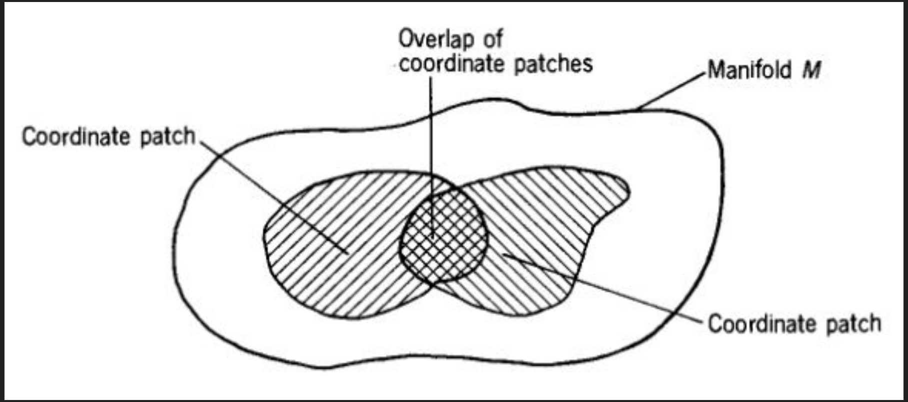
- So far we have looked at SR from a phenomenological point of view. To simplify the calculations, fully appreciate the theory and make the transition to GR, we require **tensor algebra**.

- In the following, we define tensors in **n dimensions** as objects on a geometric construct called a **manifold**. We won't go into further detail on the topological properties but assume that:

    - A n-dimensional manifold M is a **set of points**, where each point is described by a **set of coordinates** $(x^1, x^2, \ldots, x^n)$.
    - The **neighbourhood** of each point on the manifold can be **locally** mapped to a n-dimensional **Euclidean space**.

2.2 TC - coordinate patches

- A manifold cannot always be covered by a **single one-to-one correspondence** between the points and the coordinates. Instead, we define coordinate systems for multiple **coordinate patches** on M that overlap.



- **Coordinate transformations** are used to get from one patch to another.
- Behaviour of geometric quantities under transformation is central to GR.

2.3 TC - coordinate transformations

- A coordinate transformation implies that we **passively assign** a point with $(x^1, x^2, \ldots, x^n)$ the new coordinates $(x'^1, x'^2, \ldots, x'^n)$:

$$x'^a = f^a(x^1, x^2, \ldots, x^n) = f^a(x), \quad \text{for} \quad a = (1, 2, \ldots, n)$$

where fs are single-valued continuous differentiable functions.

- The **total differential** of each n of the new coordinates is then

$$\mathrm{d}x'^a = \sum_{b=1}^{n} \frac{\partial f^a}{\partial x^b}\,\mathrm{d}x^b = \sum_{b=1}^{n} \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b$$

2.4 TC - summation convention

- To write expressions in compact form, the **Einstein summation convention** is often used. It implies summation over the dimension of the manifold for repeated / dummy indices:

$$\mathrm{d}x'^{a} = \sum_{b=1}^{n} \frac{\partial x'^{a}}{\partial x^{b}}\,\mathrm{d}x^{b} \equiv \frac{\partial x'^{a}}{\partial x^{b}}\,\mathrm{d}x^{b}$$

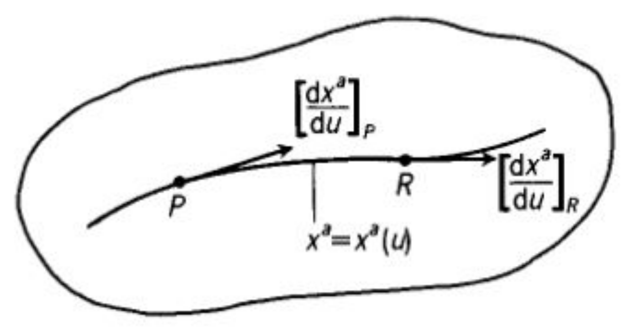
- The **Kronecker delta** is important for partial differentiation:

$$\frac{\partial x'^{a}}{\partial x'^{b}} = \frac{\partial x^{a}}{\partial x^{b}} = \delta^{a}{}_{b} = \begin{cases} 1 & \text{if} \quad a = b, \\ 0 & \text{if} \quad a \neq b. \end{cases}$$

2.5 TC - contravariant vector

- A key concept of TC is to define geometric quantities according to their **behaviour** under a **coordinate transformation**.
- A **contravariant vector** (tensor of order 1) is a set of n quantities (denoted as X^a in the x^a-coordinate system) that transforms in the following way under a change of coordinates:

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b$$



- **Example:** tangent vector to a curve $x^a = x^a$ (u), parameterised by u.

2.6 Exercise

- Show that the tangent vector t^a indeed transforms like a contravariant tensor of order 1. Use the following:

$$t^a = \frac{\mathrm{d}x^a}{\mathrm{d}u}$$

$$\mathrm{d}x'^a = \sum_{b=1}^{n} \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b \equiv \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b$$

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b$$

2.6 Exercise

- Answer:

$$t'^a = \frac{\mathrm{d}x'^a}{\mathrm{d}u} = \frac{\partial x'^a}{\partial x^b}\frac{\mathrm{d}x^b}{\mathrm{d}u} = \frac{\partial x'^a}{\partial x^b}t^b$$

2.7 TC - contravariant tensor & scalar

- We can generalise the contravariant vector definition to tensors of higher order. A **contravariant tensor of order 2** is a set of n^2 quantities (denoted as X^{ab} in x^a-coordinate system) that transforms in the following way under a change of coordinates:

$$X'^{ab} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d}X^{cd}$$

Example: product X^aY^b.

- A tensor of order 0 is a **scalar** and **invariant**, i.e., we have

$$\phi' = \phi$$

Examples: mass and charge.

2.8 TC - covariant tensors

- A **covariant tensor of order 1** is a set of n quantities (denoted as X_a in the x^a-coordinate system) that transforms as

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b$$

- This involves the **inverse transformation matrix** $\partial x^b / \partial x'^a$ and can be generalised to **higher orders** and **mixed tensors**.

- **Example**: the gradient vector of a scalar

$$\nabla \phi = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \phi, \qquad \frac{\partial \phi}{\partial x^a} \equiv \partial_a \phi \equiv \phi_{,a}$$

2.9 TC - tensor operations

- Two tensors of same type can be **added**/subtracted/multiplied:

$$X^a{}_{bc} = Y^a{}_{bc} \pm Z^a{}_{bc}, \quad X^{ab} = Y^a Z^b, \quad X^a{}_{bcd} = Y^a{}_b Z_{cd}$$

- We distinguish **symmetric** and **antisymmetric** tensors:

$$X_{ab} = X_{ba} \text{ or } X_{(ab)} = \tfrac{1}{2}(X_{ab} + X_{ba}); \quad X_{ab} = -X_{ba} \text{ or } X_{[ab]} = \tfrac{1}{2}(X_{ab} - X_{ba})$$

- We can **contract a tensor** using the Kronecker delta:

$$\delta^b{}_a X^a{}_{bcd} = X^a{}_{acd} = X^b{}_{bcd} = X_{cd}$$

2.10 Questions

- Go to www.menti.com & enter 8812 7233.
  - 1. What is the result of the scalar product $X'_a Y'^a$?
    - Contravariant tensor
    - Covariant tensor
    - Scalar
  - 2. Let's assume that two tensors satisfy $X^{ab}=Y^{ab}$ in one coordinate system. Are they equal in all other systems?
    - Yes
    - No

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    - Yes
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2.11 TC - differentiation

- **Question**: Does differentiation transform tensors into tensors?
- For a **scalar field** $\phi=\phi(x^a)$, we already mentioned that its **ordinary derivatives** are components of a **covariant tensor**:

$$\frac{\partial \phi}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial \phi}{\partial x^b}$$

- What happens if we differentiate a contravariant vector field X^a?

$$\frac{\partial X'^a}{\partial x'^c} = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b}X^b\right) = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b}X^b\right)$$
$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\frac{\partial X^b}{\partial x^d} + \frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}X^b$$

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$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\frac{\partial X^b}{\partial x^d} + \frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}X^b$$

2.12 TC - covariant derivative

- We can rewrite this by defining a new quantity $A^{a}{}_{cf}$

$$\frac{\partial X'^a}{\partial x'^c} = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\frac{\partial X^b}{\partial x^d} + \frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}\frac{\partial x^b}{\partial x'^f}X'^f$$
$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\frac{\partial X^b}{\partial x^d} + A'^a{}_{cf}X'^f$$

- This suggests that we might be able to define a **covariant derivative** in the following way:

$$\nabla_c X^a = \partial_c X^a + \Gamma^a{}_{bc} X^b \qquad \nabla_a \phi = \partial_a \phi$$

2.13 TC - affine connections

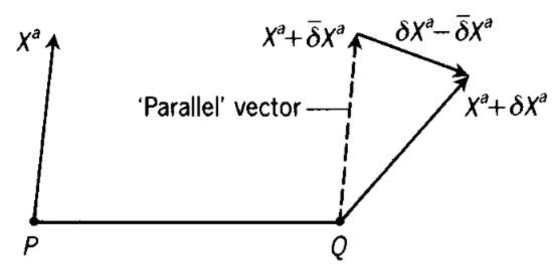
- For the covariant derivative to **transform like a tensor**, the n^3 quantities $\Gamma^a{}_{bc}$ have to satisfy the following condition

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d{}_{ef} + \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^c}$$

- The $\Gamma^a{}_{bc}$ are called **affine connections**. Because of the second, inhomogeneous term in the above transformation law, these connections are **not tensors**. Their role is to compensate for the inhomogeneous term in the vector field's partial derivative.

2.14 TC - parallel transport

- Let's consider a contravariant vector field X^a evaluated at point P and a second point Q displaced by δx^a. At Q, we can evaluate



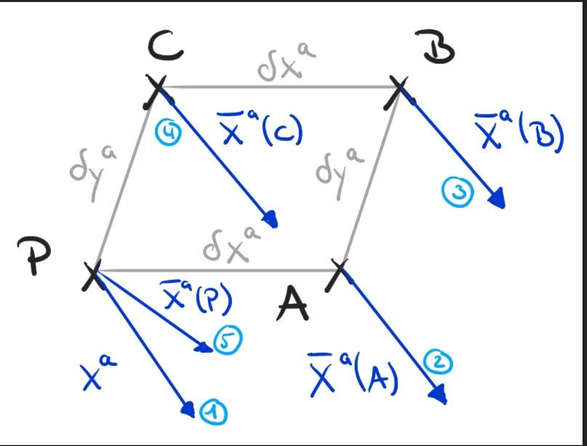
$$X^a(Q) = X^a(P) + \delta x^b \partial_b X^a = X^a(P) + \delta X^a(P)$$
$$\bar{X}^a(Q) = X^a(P) + \bar{\delta} X^a(P)$$

- δX^a is not a tensor. We construct $\bar{\delta} X^a$ so that $\delta X^a - \bar{\delta} X^a$ transforms like a tensor. This requires

$$\bar{\delta} X^a(P) = -\Gamma^a{}_{bc}(P) X^b(P) \delta x^c$$

- Connections allow us to **transport vectors** across manifolds.

2.15 TC - curvature tensor



- Imagine that we start at P and parallel transport a vector X^a along a closed path. For each path segment, we apply the previous formula to obtain the **total change** of the vector at P:

$$\bar{\delta} X^a = -\frac{1}{2} X^b R^a{}_{bcd} \left(\delta x^d \delta y^c - \delta x^c \delta y^d \right)$$

- The quantity $R^a{}_{bcd}$ is the **curvature tensor**

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed}$$

2.16 TC - affine geodesics

- Let's assume that a **curve** is parameterised by u, i.e., $x^a = x^a(u)$. Let t^a be the **tangent vector** at a point P. If we **parallel transport** the vector t^a along $x^a(u)$, then it will generally not be tangent at other points along the curve.

- If the transported vector IS tangent at any point, the curve is a so-called **geodesic curve** of our manifold and given by

$$\frac{\mathrm{d}t^a}{\mathrm{d}u} + \Gamma^a{}_{bc} t^b t^c = \frac{\mathrm{d}^2 x^a}{\mathrm{d}u^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}u} \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0$$

- Geodesics represent the **shortest path** connecting two points.

2.17 TC - metric tensor

- Consider two neighbouring points x^a and $x^a + dx^a$. If the **infinitesimal distance** between these points *ds* satisfies

$$ds^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b$$

  where g_{ab} is a **symmetric covariant tensor field** of order 2, then we call g_{ab} a **metric**. A manifold that has such a metric is called **Riemannian**. ds^2 is also known as the **line element**.

- If $g=\det(g_{ab})\neq 0$, the **inverse** g^{ab} is defined by g^{ab} is the contravariant order 2 metric tensor.

$$g_{ab}g^{bc} = \delta^c{}_a$$

2.18 TC - metric connections

- We can use these two tensors to **raise/lower indices** (two operations that are inverse to each other)

$$X^a = g^{ab}X_b, \quad X_a = g_{ab}X^b, \quad X^{ab} = g^{ac}g^{bd}X_{cd}, \quad X_{ab} = g_{ac}g_{bd}X^{cd}$$

- A Riemannian manifold has special connections, which are called **Christoffel symbols** and related to the metric as

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)$$

- The metric connections are symmetric $\Gamma^a{}_{bc}=\Gamma^a{}_{cb}$ and $\nabla_c\, g_{ab} = 0.$

2.19 TC - Riemann tensor

- For a Riemannian metric, the curvature tensor is called the **Riemann tensor** and depends on the metric and its first and second derivatives. It satisfies a number of properties, incl.

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab},$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0$$

- We call a metric **flat**, when there exists a coordinate system in which it reduces to **diagonal** form with entries ± 1 everywhere. Because g_{ab} is then constant, $\Gamma^a{}_{bc}$ and $R^a{}_{bcd}$ are zero.

2.20 TC - Ricci & Einstein tensor

- From the Riemann tensor, we construct several more important tensors. Contracting once gives the symmetric **Ricci tensor**

$$R_{ab} = R^{c}{}_{acb} = g^{cd} R_{dacb}$$

- A final contraction defines the **Ricci (curvature) scalar**:

$$R = R^{a}{}_{a} = g^{ab} R_{ab}$$

- These two define the symmetric **Einstein tensor**:

$$G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R, \quad \nabla_a G^{a}{}_{b} = 0$$

2.21 Questions

- Go to www.menti.com & enter 1252 0964.
  - 3. We parallel transport a vector around a closed path and recover exactly the same vector. It is true that …
    - the manifold is flat.
    - the Riemann tensor vanishes.
    - the total change of the vector is zero.
  - 4. A geodesic is the shortest path between two points.
    - Yes
    - No

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Summary I:

Covered today: special relativity, tensor calculus

- Einstein combined earlier results to develop a **new theory of (special) relativity** based on two postulates: i) all inertial frames are equivalent, ii) the speed of light is constant.
- In SR, the laws of physics are invariant under **Lorentz transformations**, which couples space & time into **spacetime**.
- Special relativity **extends Newtonian physics** to those cases where speeds are close to that of light (but gravity negligible).

Summary II:

Covered today: special relativity, tensor calculus

- To simplify the SR formalism and eventually appreciate the beauty of GR, we make use of **tensor calculus**. We will use tensors to write equations in **coordinate independent** form.
- Tensors are objects satisfying certain properties under **coordinate transformations**. We distinguish scalars (mass), contravariant (tangent vector) and covariant (gradient) tensors.
- Using the formalism, we can encode information about a manifold's **curvature** and determine **geodesics** and **distances**.